Compensating Wage Differentials in Stable Job Matching Equilibrium

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Abstract

This paper studies a stable job matching equilibrium and the implicit pricing of non-wage job characteristics. It departs from the previous literature by allowing worker heterogeneity in productivity instead of preferences, giving rise to a double transaction problem in a hedonic model. We show explicitly how wage differences across jobs can be decomposed into compensating wage differentials for non-wage job characteristics and differences in worker productivity. We also derive sufficient conditions for an assortative job matching and a stable matching condition in a model with continuous agent types. Empirical evidence from the U.S. Census and job amenity data from the Dictionary of Occupational Titles strongly supports our theory.

1 Introduction

The theory of equalizing differences has remained one of the most fundamental value theories in economics since Adam Smith's classic discussion in "The Wealth of Nations." Rosen (1974) develops a theory of equalizing differences for a commodity market and shows that implicit markets arise for differentiated products with hedonic

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prices adjusting so that all markets clear. Rosen (1986) applies this theory to the labor market and finds that equilibrium wage differentials reflect workers' willingness to pay for job characteristics. His theory demonstrates how non-wage characteristics of a job are valued and how workers and jobs are sorted when workers' preferences are heterogeneous.

However, in an environment where workers are heterogeneous in their productivity, instead of preferences, the values of job characteristics and worker productivity, and the nature of sorting are not well understood. The difficulty in examining the problem arises from double transactions: workers sell their productivity characteristics to firms, while the firms implicitly sell job characteristics to the workers. Double transactions are inherent in labor markets and absent in commodity markets, because consumers do not sell their characteristics to firms. In other words, the characteristics of consumers do not directly affect the profit of firms. To the contrary in a labor market, not only the wage, but also workers' characteristics affect the employer's profit. This implies that the wage difference between jobs does not only reflect the compensating wage differentials for the differences in job characteristics, but also reflects the difference in the productivity of the workers occupying the jobs. This double transaction also complicates the sorting mechanism. The established results in the optimal assignment literature are often intended for the analysis of the commodity market and are not readily applicable to a labor market where the double transaction problem plays an important role.

Given that productivity is arguably the most important characteristic of workers, solving the double transaction problem is an important step toward a thorough understanding of how workers find satisfactory employment and how all the various job and worker characteristics are valued in the labor market. Rosen noted the importance of workers' productivity heterogeneity in his article (pp. 688, 1986), pointing out

> "...On the theoretical side of these questions, much more attention must be paid to the value of workers' productivity characteristics and the nature of sorting and selection in those dimensions...."

The objective of this paper is to solve this double transaction problem and provide insight into how non-wage job characteristics are implicitly valued in the market and how workers and jobs are matched. Taking advantage of the main prediction of the theory, we also estimate compensating wage differentials separately from the worker productivity effect.

In the model workers are heterogeneous in their productivity and jobs differ in terms of their characteristics. Workers receive utility not only from a wage but also from the characteristics of the job that they acquire. Moreover, good characteristics may increase the marginal utility of consumption or the marginal productivity of a worker. When this complementarity exists, assortative matching occurs as the unique matching pattern. We derive the sufficient condition for assortative matching in a continuous model with a continuum of firms and workers and non-transferable payoff functions. In assortative matching, matching between workers and jobs takes place through ranking: the t th best worker occupies the t th best job. The wage difference between two jobs reflects a compensating wage differential and the difference in productivity between workers occupying the jobs. On the one hand, the former lowers the wage of a better job, because a worker is willing to accept a lower wage in exchange for better job characteristics. On the other hand, the latter raises the wage of a better job, because the worker occupying the better job is more productive. Hence, the observed wage gap between two occupations does not necessarily reflect compensating wage differentials.

The main finding of our theoretical analysis is that the worker-job matching pattern drives the size of the worker productivity effect that appears in the wage. A matching function fully describes who is matched to which job, and its shape depends on the distributions of worker productivities and job characteristics. Suppose that the distribution of worker productivity becomes more dispersed for an exogenous reason, while the distribution of job characteristics remains the same. This exogenous change increases the gap in labor productivity between the t th and $t+1$ th best jobs because workers and jobs are assortatively matched. Thus, the wage difference between the t th and $t+1$ th best job reflects a large worker productivity gap in addition to the compensating wage differentials. In contrast, when worker productivity exhibits little variation, the productivity of the worker occupying the t th best job is about the same as that of the worker occupying the $t+1$ th job. In this case, the wage gap between the t and $t+1$ th best jobs mostly reflects compensating wage differentials. When the worker productivity effect dominates the compensating wage differentials, the observed wage gap across jobs seem contradictory to the theory of equalizing differences because the observed wage increases (instead of decreases) with job characteristics.

We find strong evidence that supports our theory from the U.S. 2000 Census and job amenity data from the Dictionary of Occupational Titles (DOT). First, workers and jobs are assortatively matched in all the U.S. states. Second, when a state's matching function indicates a large productivity gap between jobs, the return to job amenity is also positive and large when controlling for standard worker characteristics. In contrast, in a state where its matching function shows a small productivity gap between jobs, the return to job amenity is negative, which is consistent with the notion of equalizing differences. Using the first order condition for the stable matching equilibrium, we then counterfactually calculate the return to job amenity in the absence of the worker productivity effect, which is exactly the compensating wage differential for the job amenity. Our estimate indicates that an increase of the job amenity by one standard deviation decreases the wage by 8%, while a naive OLS estimate indicates a wage increase of 2%. The difference is 10 percentage points, and we find this bias quantitatively significant.

The rest of the paper is structured as follows. Section 2 reviews the related literature. We develop our theory in Section 3. Section 3 provides the empirical evidence that supports our theory. We conclude in Section 5.

2 Literature Review

Rosen (1986) shows how compensating wage differentials differ across workers when they are heterogeneous in preferences, but homogeneous in productivity. This is similar to hedonic prices for differentiated products in the commodity market. Scotchmer (1985) applies the theory to the housing market, and Kanemoto (1988) studies hedonic prices for public projects. More recently Ekeland (2010) and Chiappori et al. (2010) undertake hedonic equilibrium analysis in a more general matching model that is intended for various commodity markets. In commodity markets, prices are different for differentiated products. The price differentials reflect the buyer's marginal willingness to pay for a better product. One key feature of these models is the lack of double transactions, i.e., the characteristics of consumers do not directly affect the profit of firms. These models are not directly applicable to a labor market unless double transactions are excluded, which implies that workers are homogeneous in productivity.

We also contribute to the optimal assignment literature. Shapley and Shubik (1972), Gretsky et al. (1992, 1999), and Kelso and Crawford (1982) study how agents on one side of the market are matched with agents on the other side. However, their assignment mechanisms are intended for the analysis of the commodity market. On the other hand, Teulings (1995) and Shimer (2005) study the assignment of workers to firms when workers and firms are heterogeneous in productivity. However, non-wage job characteristics do not affect the utility of workers in their model. Sattinger (1997) provides a closed-form analysis of compensating wage differentials based on a constant elasticity of substitution utility function and a log-normal distribution of worker productivity. However, the equilibrium wage function is derived with an ad hoc assumption on its form, given the partial equilibrium approach, without modeling the firm's production function. We derive a sufficient condition for assortative matching when double transactions exist. In addition, we provide a closed-form analysis under various environments in the Appendix.

The optimal assignment literature has emphasized the importance of when to expect assortative matching under non-transferable utility. Recently, Legros and Newman (2007) show that the condition called "generalized increasing differences" are sufficient for assortative matching with finite numbers of agents on each side with non-transferable utility. This condition is characterized in terms of the properties of the indirect payoff function that defines the agent's maximum payoff as a function of his type, his partner's type, and the partner's payoff. The sufficient condition for assortative matching derived in our paper is also derived under non-transferable utility but differs in that the sufficient condition for assortative matching is characterized in terms of the properties of the primitive payoff functions with a continuum of agents on each side. The major advantage of a continuous model is its usefulness in analyzing compensating wage differentials. In a continuous model, the relationship between the equilibrium wage and the equilibrium matching pattern is clearly characterized by a first-order differential equation. This differential equation provides a new estimation strategy for equilibrium wage, equilibrium matching, and compensating wage differentials.

There is a huge empirical literature examining compensating wage differentials. Empirical tests of compensating wage differentials in the labor market have often been found to be inconclusive, mixed, and even counterintuitive (i.e., opposite signs) compared to the theory. Various empirical reasons have been posited for the difficulty in estimating equalizing differences in the labor market. Among them, unobserved heterogeneity (Hwang et al. 1992), omitted variables (Brown 1980, Lucas 1977), measurement errors (Duncan and Holmlund 1983) and an unappealing linear approximation (Ekeland et al. 2004) have been considered as sources of the counter-intuitive results. Their theoretical background is Rosen (1986), which assumes homogeneity in worker productivity. The empirical papers implicitly assume that labor markets are segmented by skill levels and that workers are homogeneous in productivity within a given market segment. The effects of worker skills on wages are controlled for by observed worker characteristics and instrumental variables if available. Unlike previous papers, we explicitly show how worker productivity and compensating wage differentials appear in the wage equation. Taking advantage of the theory, we estimate compensating wage differentials separately from the productivity effect. Our approach provides a useful alternative to the instrumental variable approach when the assumptions required to justify instrumental variables are not plausible.

Hwang et al. (1998) and Lang and Majumdar (2004) study compensating wage differentials in the presence of search frictions in the labor market. They show that the equilibrium distribution of wages across different jobs may not exhibit compensating wage differentials when workers are homogeneous in their productivity. Our model is relevant for an environment in which workers are heterogeneous in productivity and labor market frictions do not exist.

3 Theory

The key intuition of equalizing differences in the labor market can be explained without heterogeneity in worker productivity. Consider the following model. Firm x has a job with characteristic $x \geq 0$. Throughout the paper, let $F(x)$ be the measure of firms whose job characteristics are no more than x. There is a continuum of workers in the market with a total measure of one. The total measure of firms is $T \geq 1$ and hence all workers who actively look for jobs are hired in the competitive equilibrium. Let the utility function $u(c,x)$ represent a worker's preferences over the consumption good c that he/she explicitly purchases and the characteristic x of the job that he/she occupies. Assume that worker preferences are monotonic, so that the marginal utilities, $u_c(c,x)$

and $u_x(c,x)$, of the consumption good and the job characteristic are both positive at all $(c,x) \in \mathbb{R}^2_+$. Let workers' exogenous non-earned income be normalized to zero. Then, if a worker's wage is w, he/she purchases w units of the consumption good ($c = w$).

Let $\hat{y}$ be every worker's labor productivity characteristic. Given the homogeneity in worker productivity, let the firm's production function be characterized by $f(x,\hat{y})$. There are potentially more firms than workers in the market, and a worker's utility is increasing in the job characteristic, so the set of firms who hire workers in equilibrium is $\{x \in \mathbb{R}_+ : x \geq \underline{x} \text{ with } T - F(\underline{x}) = 1\}$. Let $w(x)$ be the equilibrium wage paid to the worker who gets job x. In equilibrium, firm $\underline{x}$ has zero profit, otherwise those firms whose job characteristics are lower than but arbitrarily close to $\underline{x}$ would make wage offers to the worker to attract him/her. Therefore, $w(\underline{x}) = f(\underline{x},\hat{y})$. Workers are homogeneous in their productivity, so every worker must have the same equilibrium utility, $u(w(x),x) = u^*$, for all $x \geq \underline{x}$. Since firm $\underline{x}$ pays its total output to its worker as a wage, each worker's equilibrium wage is determined by $u(w(x),x) = u(w(\underline{x}),\underline{x})$ for all $x \geq \underline{x}$. Worker utility is increasing in the job characteristic, so the less preferred job must be compensated by a higher wage in equilibrium in order to equalize the workers' utilities, i.e., $w(x) > w(x')$ if $x < x'$.

Therefore, the observed wage should be negatively related to the job characteristic. This is called an equalizing difference in the labor market or a compensating wage differential as explained in Rosen (1986). This relationship is solely driven by the workers' monotonic preferences. It does not depend on the nature of the job characteristic in the production technology or on other properties of the workers' preferences.

However, double transactions occur in the labor market when a firm hires a worker: the worker sells his/her productivity characteristic to the firm, and the firm sells the job characteristic to the worker. Therefore, the equilibrium wage will reflect both prices: one for the worker's productivity characteristic and the other for the job characteristic. To study how these two prices determine the equilibrium wage, we set up a competitive equilibrium in a labor market where workers' productivity characteristics and the firms' job characteristics are heterogeneous. Let $y \geq 0$ denote a worker's productivity characteristic. We assume that workers all differ in their productivity so that worker y denotes the worker with productivity y. Let $G(y)$ be the measure of workers whose characteristics are no more than y, where the total measure is one. We also assume that firms all differ in the characteristics of their jobs so that firm x denotes the firm that has a job with characteristic x. Let $f(x,y)$ denote a firm's production function. We assume that the production function is increasing in y and non-decreasing in x: $f_y(x,y) > 0$ and $f_x(x,y) \geq 0$ at all (x,y). If $f_x(x,y) = 0$, then the job characteristic only has a private value. If $f_x(x,y) > 0$, then it has a common value.

We construct a competitive equilibrium with the notion of stable matching where there are no alternative pairs of firms and workers who, by matching and transferring wage, can make themselves strictly better off. Stable matching can be characterized by the market matching function and the market wage function. Let $m(x)$ be the market matching function that specifies the productivity characteristic of the worker who works for firm x in equilibrium. $m(x) = \emptyset$ means that firm x does not hire any workers. Let $w(x)$ be the market wage function that specifies the wage firm x pays to its worker.

Given the expectations on $\{m(\cdot), w(\cdot)\}$, the worker believes that the equilibrium profit for firm x is $f(x,m(x)) - w(x)$. Therefore, worker y believes that he/she can work for firm x if he/she is willing to accept wage w such that $f(x,y) - w \geq f(x,y(x)) - w(x)$. Worker y wants to work for firm x at wage w if (x,w) solves problem (W1) below:

$$\text{(W1)} \qquad \begin{aligned} &\max_{(x,w)} u(w,x) \\ &\text{subject to } f(x,y) - w \geq f(x,m(x)) - w(x). \end{aligned}$$

Let $\tilde{x}(y)$ be the characteristic of the job that worker y would get (i.e., $m(\tilde{x}(y)) = y$) if job matching took place according to the market matching function. Given $\{m(\cdot), w(\cdot)\}$, the firm believes that $u(w(\tilde{x}(y)), \tilde{x}(y))$ is the equilibrium utility that the worker would receive if he/she worked for firm $\tilde{x}(y)$ at wage $w(\tilde{x}(y))$. Therefore, firm x must pay wage w satisfying $u(w,x) \geq u(w(\tilde{x}(y)), \tilde{x}(y))$ if it wants to hire worker y. Firm x wants to hire worker y at wage w if (y,w) solves

$$\text{(F1)} \qquad \begin{aligned} &\max_{(y,w)} \{f(x,y) - w\} \\ &\text{subject to } u(w,x) \geq u(w(\tilde{x}(y)), \tilde{x}(y)). \end{aligned}$$

Worker y and firm x match at wage w when (x,w) solves problem (W1) for worker y, and (y,w) solves problem (F1) for firm x . In a stable matching equilibrium, the characteristic of the worker who works for firm x is exactly the same as the one, $m(x)$, specified in the market matching function, and the wage is exactly the same as the one, $w(x)$, specified in the market wage function. Therefore, expectations are fully realized in a stable matching equilibrium.

Definition 3.1 *Given distributions of the job characteristic and the workers' productivity characteristics, $\{m(\cdot), w(\cdot)\}$ is a stable matching equilibrium in which for all y, worker y works for firm x at wage $w(x)$ if (i) $(x, w(x))$ is a solution to problem (W1) for worker y; (ii) $(y, w(x))$ is a solution to problem (F1) for firm x; and (iii) $y = m(x)$.*

The set $\{m(\cdot), w(\cdot)\}$ that satisfies conditions (i)-(iii) in Definition 3.1 leads to stable job matching because it induces no pairs of firms and workers who, by matching and transferring wage, can make themselves strictly better off. We first examine the set of firms who hire workers in a stable matching equilibrium.

Lemma 3.1 *Any firm who hires a worker in a stable matching equilibrium has a job characteristic x that is no less than $\underline{x}$, where $\underline{x}$ satisfies $T - F(\underline{x}) = 1$. Furthermore, $w(\underline{x}) = f(\underline{x}, m(\underline{x}))$.*

The equilibrium wage for the worker in the worst match is uniquely determined because there are potentially more firms than workers in the market so that competition among firms drives up the equilibrium wage for the worker in the worst match to his total product. Otherwise, the worker with $y(\underline{x})$ and a firm, with a characteristic very similar to $\underline{x}$, that does not fill its job can agree on a wage that makes them both better off. The fact that there are potentially more firms than workers pins down the wage for the worker in the worst match and subsequently the equilibrium wage levels in all matches.[1]

Next, we show that a worker's utility is increasing in his/her productivity characteristic in a stable matching equilibrium.

Lemma 3.2 *The worker's equilibrium utility, $u(w(\tilde{x}(y)), \tilde{x}(y))$, is increasing in his/her productivity characteristic y.*

The increasing property of a worker's equilibrium utility with respect to his/her productivity characteristic comes from the monotonicity of worker preferences and the monotonicity of the production function in the worker's productivity characteristic. When workers are homogeneous in their productivity, every worker receives the same

[1]If the measure of firms is equal to the measure of workers, then the equilibrium wage function may not be unique because the wage in the worst match is not uniquely determined.

equilibrium utility level as shown in section 2. In this case, a less preferred job characteristic must be compensated by a higher wage in order to equalize the equilibrium utility across workers. However, when there is heterogeneity in workers' productivity, a worker's equilibrium utility is increasing in his/her productivity. Because various combinations of wages and job characteristics can increase a worker's utility, the relationship between the equilibrium wage and the job characteristic is not clear, contrary to the sharp characterization of equalizing differences in a labor market without heterogeneity in worker productivity. The next two sections examine the determinants of the stable job matching pattern between the worker's productivity and the job characteristic, and the implications of heterogeneity in worker productivity and the stable matching pattern on the relationship between the equilibrium wage and the job characteristic.

3.1 Matching without Matching Patterns

We first examine when the part of the wage that is attributed to the worker's productivity can be independently decomposed from the compensating wage differentials in the equilibrium wage. Suppose that workers' preferences are quasilinear with respect to consumption, so that they are given by the utility function $u(c,x) = c + \phi(x)$, where $\phi(\cdot)$ is increasing in x. In this case, the marginal utility of consumption and the marginal utility of the job characteristic are independent , and the job characteristic is substitutable for consumption. Suppose that the job characteristic has only a private value in the sense that it does not affect the worker's productivity. Write the production function as $f(y)$ for this case. Theorem 3.1 below fully characterizes a stable matching equilibrium when the workers' preferences are quasilinear and the job characteristic has only a private value.

Theorem 3.1 *Suppose that the workers' preferences are quasilinear and the job characteristic has only a private value.* $\{m(\cdot), w(\cdot)\}$ *is a stable matching equilibrium if (and only if):*

1. $m(\cdot)$ *is any arbitrary matching function that satisfies Lemma 3.1 and*

2. *given any matching function* $m(\cdot)$, $w(\cdot)$ *satisfies*

$$w(x)+\phi(x)=f(m(x))+\phi(\underline{x}) \tag{3.1}$$

for all $x \geq \underline{x}$.

Furthermore, the equilibrium profit for firm x *is* $\phi(x)-\phi(\underline{x})$ *for all* $x \geq \underline{x}$.

The key intuition behind Theorem 3.1 is that the equilibrium profit difference for any two different firms is the same as the utility difference associated with their job characteristics: for any $x, \acute{x} \geq \underline{x}$ such that $x \neq \acute{x}$,

$$f(m(x))-w(x)-[f(m(\acute{x}))-w(\acute{x})]=\phi(x)-\phi(\acute{x}). \tag{3.2}$$

Suppose that (3.2) does not hold. Without loss of generality, consider the case where the right-hand side of (3.2) is bigger, so we have

$$f(m(x))-w(x)<f(m(\acute{x}))-w(\acute{x})+\phi(x)-\phi(\acute{x}). \tag{3.3}$$

Note that firm x could hire the worker who works for firm $\acute{x}$ if it offered a wage w that ensures the worker's utility $w+\phi(x)$ at least as high as $w(\acute{x})+\phi(\acute{x})$ (i.e., $w = w(\acute{x})-\phi(x)+\phi(\acute{x})$). It implies that the right-hand side of (3.3) is the maximum profit that firm x could receive from hiring worker $m(\acute{x})$. Because this is greater than the profit for firm x generated by hiring worker $m(x)$ at $w(x)$, firm x and worker $m(\acute{x})$ are both better off at a wage just slightly lower than $w(\acute{x})-\phi(x)+\phi(\acute{x})$.

Applying (3.2) to $\acute{x}=\underline{x}$ with $w(\underline{x})=f(m(\underline{x}))$, we can deduce the equilibrium profit for firm x for $x \geq \underline{x}$:

$$f(m(x))-w(x)=\phi(x)-\phi(\underline{x}) \tag{3.4}$$

for all $x \geq \underline{x}$. Equation (3.4) shows that the firm is the fixed residual claimant in equilibrium regardless of which worker it hires in equilibrium. Because the marginal utility of consumption and the marginal utility of the job characteristic are independent of each other, the worker has no preferences over the composition of the compensation package This means that it does not matter for the worker whether he/she receives utility in the form of the wage or the job characteristic as long as the overall compensation package correctly reflects the output produced by him/her.

These points are clear if we compare two different equilibria $\{m(\cdot), w(\cdot)\}$ and $\{\hat{m}(\cdot), \hat{w}(\cdot)\}$. First of all, consider the equilibrium profit for firm x for $x \geq \underline{x}$. According to equation (3.4), its profits are

$$\begin{aligned} f(m(x)) - w(x) &= \phi(x) - \phi(\underline{x}), \\ f(\hat{m}(x)) - \hat{w}(x) &= \phi(x) - \phi(\underline{x}) \end{aligned}$$

in stable matching equilibria $\{m(\cdot), w(\cdot)\}$ and $\{\hat{m}(\cdot), \hat{w}(\cdot)\}$ respectively. Hence the equilibrium profit for firm x is the same as $\phi(x) - \phi(\underline{x})$ across any two different stable matching equilibria.

Secondly, consider the equilibrium utility for worker y. Assume that worker y is hired by firm x in the stable matching equilibrium $\{m(\cdot), w(\cdot)\}$ but by firm x' in the stable matching equilibrium $\{\hat{m}(\cdot), \hat{w}(\cdot)\}$. It means that $y = m(x) = \hat{m}(x')$. According to equation (3.4), the equilibrium utilities for worker y are

$$\begin{aligned} u(w(x), x) &= w(x) + \phi(x) = f(m(x)) - \phi(\underline{x}), \\ u(\hat{w}(x'), x') &= \hat{w}(x') + \phi(x') = f(\hat{m}(x')) - \phi(\underline{x}) \end{aligned}$$

in stable matching equilibria $\{m(\cdot), w(\cdot)\}$ and $\{\hat{m}(\cdot), \hat{w}(\cdot)\}$ respectively. Because $f(m(x)) = f(\hat{m}(x')) = f(y)$, the equilibrium utility for worker y is the same as $f(y) - \phi(\underline{x})$ across any two different stable matching equilibria.

The profit and utility comparison across two different stable matching equilibria shows that given *any* matching function, the wage function is properly adjusted so that the equilibrium utility for worker y is always $f(y) - \phi(\underline{x})$ and the equilibrium profit for firm x is always $\phi(x) - \phi(\underline{x})$ for $x \geq \underline{x}$ in any stable matching equilibrium.[2] Because any matching function is an equilibrium matching function given a proper wage function, a systematic correlation is not expected between the worker's productivity characteristic and the firm's job characteristic in a stable matching equilibrium.[3] From the firm's equilibrium profit, we can derive the worker's equilibrium utility as (3.1) and also the equilibrium wage equation as follows:

[2] If $x < \underline{x}$, then firm x receives zero profits in any stable matching equilibria.

[3] This is reminiscent of the Modigliani-Miller theorem (Modigliani and Miller 1958) in corporate finance, which shows that the value of a firm is unaffected by how that firm is financed. It does not matter if the firm's capital is raised by issuing stock or selling debt.

Corollary 3.1 *The equilibrium wage equation is then*

$$w(x) = \phi(\underline{x}) + f(m(x)) - \phi(x) \tag{3.5}$$

for all $x \geq \underline{x}$.

When no systematic correlation is observed between the job characteristic and the worker's productivity characteristic, we can indeed isolate the effect of the compensating wage differentials in the coefficient of the job characteristic: it is negative as the theory of equalizing differences predicts. The part of the wage that is attributed to the worker's productivity is independently captured in the second term on the right-hand side of equilibrium wage equation (3.5).

3.2 Assortative Matching

It is often observed that high productivity workers are matched to firms with better jobs. This implies that the equilibrium matching function $m(\cdot)$ is increasing in x. This outcome is called positively assortative matching. We first study when positively assortative matching is expected in equilibrium. In order to study assortative matching, we must first introduce some notation. Suppose that X is a partially ordered set with the transitive, reflexive and antisymmetric order relation $\geq$.[4] For any x and x' in X , let $x \vee x'$ denote the least upper bound (join) of x and x', and let $x \wedge x'$ denote the greatest lower bound (meet). If $X \subseteq \mathbb{R}^n$, then the join of x and x' is the component-wise maximum and the meet is the component-wise minimum. The set X is a lattice if, for any x and x' in X, thier joint and meet exist as elements of X. Consider a subset X of $\mathbb{R}^n$ that is a lattice with the ordering relation $\geq$ such that $x = (x_1, \ldots, x_n) \geq x' = (x'_1, \ldots, x'_n)$ if $x_i \geq x'_i$ for all $i = 1, \ldots, n$. Any real-valued function $g : X \to \mathbb{R}$ with a lattice $X \subseteq \mathbb{R}^n$ is supermodular (equivalently x_i and x_j are complementary for all i and j such that $i \neq j$) if

$$g(x) + g(x') \leq g(x \vee x') + g(x \wedge x') \tag{3.6}$$

[4] See Topkins (1998) for a comprehensive review on the theory of lattice, supermodularity, and their applications in economics. An order relation is reflexive if $x \geq x$ for all $x \in X$ and antisymmetric if $x \geq x'$ and $x' \geq x$ implies that $x = x'$.

for all x and x'. The function g is said to be strictly supermodular (equivalently x_i and x_j are strictly complementary for all i and j such that $i \neq j$) if, for all unordered x and x', (3.6) holds with strict inequality. When the function g is twice differentiable, supermodularity is equivalent to $g_{ij} = \partial^2 g/\partial x_i \partial x_j \geq 0$ for all i and j such that $i \neq j$ and strict supermodularity is equivalent to $g_{ij} = \partial^2 g/\partial x_i \partial x_j > 0$ for all i and j such that $i \neq j$.

Supermodularity is quite natural in many cases.[5] The separable functions are supermodular and supermodularity is sufficiently general to allow for a privately valued job characteristic (i.e., it does not affect the worker's productivity). Note that, although they are supermodular, separable functions are not strictly supermodular.

To illustrate sufficient conditions for a positively assortative stable job matching, we first introduce an additional notation $w^*(x,u)$. This denotes the minimum wage that firm x must offer to the worker in equilibrium to give him/her a utility level at least as high as u. Therefore, the equation $u(w^*(x,u),x) - u = 0$ defines the implicit function $w^*(x,u)$. Taking the cross partial derivative of $u(w^*(x,u),x) - u = 0$ with respect to x and u yields

$$w^*_{xu}(x,u) = -\frac{u_{cc}(w,x)}{u_c(w,x)} w^*_x(x,u) w^*_u(x,u) - \frac{u_{cx}(w,x)}{u_c(w,x)} w^*_u(x,u). \tag{3.7}$$

Given the utility level u, if the job characteristic increases, the wage must decrease: $w^*_x(x,u) < 0$. If the utility level u increases given the job characteristic, the wage must increase: $w^*_u(x,u) > 0$. Because the worker's utility function is increasing in consumption, we have $u_c(w,x) > 0$. Suppose that the worker's utility function is concave in consumption so that $u_{cc}(w,x) \leq 0$. Given the signs of the above derivatives, if the worker's utility function is supermodular (i.e., $u_{cx}(w,x) \geq 0$), then we have $w^*_{xu}(x,u) \leq 0$ at each (x,u). When the cross partial derivative is non-positive, the function is called submodular.

Consider two pairs of utility levels and job characteristics, u_H and u_L $(u_H > u_L)$

[5] In our context, the supermodular utility functions include, among many others, (a) substitutable consumption and job characteristic: $u(c,x) = \alpha c^a + \beta x^b$ with $\alpha \geq 0$ and $\beta \geq 0$, (b) perfect complements: $u(c,x) = \min\{\alpha c, \beta x\}$ with $\alpha \geq 0$ and $\beta \geq 0$, (c) Cobb-Douglas utility function: $u(c,x) = \beta c^t x^s$ with $\beta \geq 0$, $t \geq 0$, $s \geq 0$, and (d) constant elasticity of substitution utility function: $u(c,x) = (\alpha c^\rho + \beta x^\rho)^{1/\rho}$ with $\alpha \geq 0$, $\beta \geq 0$, $\rho \leq 1$. The classes of utility functions, (c) and (d), are strictly supermodular if the parameter restrictions hold with strict inequality. Examples of common supermodular production functions are similar.

and x_H and x_L $(x_H > x_L)$. The non-positive sign of $w^*_{xu}(x,u)$ is equivalent to

$$w^*(x_L,u_H) - w^*(x_L,u_L) \geq w^*(x_H,u_H) - w^*(x_H,u_L). \tag{3.8}$$

If the worker's utility function is strictly supermodular (i.e., $u_{cx}(w,x) > 0$), $w^*_{xu}(x,u)$ is negative and hence (3.8) holds with strict equality. The interpretation of (3.8) is natural. Suppose that two firms, firm x_H and firm x_L, consider hiring one of two workers, one of whom is supposed to receive u_H because his/her productivity is higher and the other who is supposed to receive u_L. Each firm must offer a higher wage to the worker with higher productivity than it offers to the worker with lower productivity. (3.8) implies that the differential between the wages that firm x_L would offer to the workers (the left-hand side of (3.8)) is at least as big as the differential between the wages that firm x_H would offer to the workers (the left-hand side of (3.8)). This is the key property behind Theorem 3.2.

Theorem 3.2 *The market matching function $m(\cdot)$ is increasing if the utility function is concave in consumption (i.e., $u_{cc}(c,x) \leq 0$ at each (c,x)) and either condition (a) or condition (b) is satisfied:*

(a) $u(\cdot,\cdot)$ is supermodular and $f(\cdot,\cdot)$ is strictly supermodular

(b) $u(\cdot,\cdot)$ is strictly supermodular and $f(\cdot,\cdot)$ is supermodular

Proof. We prove Theorem 3.2 by contradiction. Consider two firms, firm x_H and firm x_L, such that $x_H > x_L$, and their workers, worker $m(x_H)$ and worker $m(x_L)$. Contrary to positively assortative matching, suppose that $m(x_H) < m(x_L)$ so that firm x_L hires the worker with higher productivity. Because a worker's equilibrium utility is increasing in his/her productivity by Lemma 3.2, we have $u(w(x_L),x_L) > u(w(x_H),x_H)$, where the left-hand side is the equilibrium utility for worker $m(x_L)$ and the right-hand side is the equilibrium utility for worker $m(x_H)$. Let $u_H = u(w(x_L),x_L)$ and $u_L = u(w(x_H),x_H)$. The market wages, $w(x_L)$ and $w(x_H)$, themselves are the minimum wages that firm x_L and firm x_H must offer in order to induce utility levels for the worker which are at least as high as u_H and u_L, respectively, given their job characteristics. Therefore, we have $w(x_L) = w^*(x_L,u_H)$ and $w(x_H) = w^*(x_H,u_L)$.

Because $w^*(x_L,u_L)$ is the minimum acceptable wage that firm x_L must offer worker $m(x_H)$ (i.e., $u_L = u(w^*(x_L,u_L),x_L)$) and it is not profitable for firm x_L to hire worker

$m(x_H)$ in a stable matching equilibrium, we have $f(x_L, m(x_L)) - w(x_L) \geq f(x_L, m(x_H)) - w^*(x_L, u_L)$. Because $w(x_L) = w^*(x_L, u_H)$, this inequality yields

$$f(x_L, m(x_L)) - f(x_L, m(x_H)) \geq w^*(x_L, u_H) - w^*(x_L, u_L). \tag{3.9}$$

When the worker's utility function is supermodular, (3.8) and (3.9) imply that

$$f(x_L, m(x_L)) - f(x_L, m(x_H)) \geq w^*(x_H, u_H) - w^*(x_H, u_L). \tag{3.10}$$

If the worker's utility function is strictly supermodular, (3.10) holds with strict inequality because (3.8) holds with strict inequality. If the production function is supermodular, we have

$$f(x_H, m(x_L)) - f(x_H, m(x_H)) \geq f(x_L, m(x_L)) - f(x_L, m(x_H)). \tag{3.11}$$

If the production function is strictly supermodular, (3.11) holds with strict inequality.

If one of conditions (a) and (b) holds, one of (3.10) and (3.11) holds with strict inequality. Therefore, (3.10) and (3.11) yield

$$f(x_H, m(x_L)) - f(x_H, m(x_H)) > w^*(x_H, u_H) - w^*(x_H, u_L).$$

Because $w^*(x_H, u_L) = w(x_H)$, rearranging the inequality above yields

$$f(x_H, m(x_L)) - w^*(x_H, u_H) > f(x_H, m(x_H)) - w(x_H).$$

This shows that the profit for firm x_H from hiring worker $m(x_L)$ at wage $w^*(x_H, u_H)$ is strictly higher than the profit from hiring worker $m(x_H)$. Therefore, there is a wage that is mutually beneficial to firm x_H and worker $m(x_L)$. Because of this contradiction, $m(\cdot)$ must be an increasing function. ■

These sufficient conditions encompass models with non-transferable utility. Because condition (a) implies that the worker's productivity characteristic and the firm's job characteristic are strictly complementary in the production function, it is intuitive to have positively assortative stable matching between them as long as the job characteristic does not decrease the marginal utility of consumption. Condition (b) shows that the strict complementarity between consumption and the job characteristic in the worker's utility function induces a positively assortative stable job matching between the worker's productivity characteristic and the firm's job characteristic even if they are not strictly complementary in the production function. To illustrate this point, write the production function as $f(y)$ so that the job characteristic has only a private value. If the worker's utility function is strictly supermodular, then (3.10) alone induces

$$f(m(x_L)) - w^*(x_H, u_H) > f(m(x_H)) - w(x_H),$$

where $w(x_H) = w^*(x_H, u_L)$. If stable job matching is not positively assortative (i.e., $m(x_H) < m(x_L)$), the profit firm x_H receives when hiring worker $m(x_L)$ at wage $w^*(x_H, u_H)$ is strictly higher than the profit generated from hiring worker $m(x_H)$ at wage $w(x_H)$. Because of this contradiction, stable job matching must be positively assortative. This shows how stable job matching can be uniquely positively assortative between the firm's job characteristic and the worker's productivity even without direct strict complementarity.

Our results differ from Legros and Newman (2007) in that the sufficient conditions for assortative matching are derived in terms of properties of the primitive payoff function in environments with a continuum of agents on each side, while Legros and Newman (2007) assume a finite number of agents. The major advantage of a continuous model is its usefulness in analyzing compensating wage differentials because the equilibrium wage function can be characterized by the first-order differential equation in (3.12) together with the equilibrium matching function.[6]

3.3 Discussion

Positively assortative job matching implies that the part of the equilibrium wage that is attributed to the worker's productivity can also be explained by the job characteristic. First-order conditions for problems (W1) and (F1) for worker $y(x)$ and firm x,

[6]Gretsky et al. (1992, 1999) extended the finite version (Shapley and Shubik 1972) of optimal assignment problems in the goods market with transferable utility to models with a continuum of buyers and sellers. However, the continuous version in Gretsky et al. (1992, 1999) is not appropriate for analysis of compensating wage differentials because their focus is the goods market. They also did not cover non-transferable utility which is often observed in the worker's utility. Also, the continuous hedonic models in Ekeland (2010) and Chiappori et al. (2010) are not appropriate for the same reason.

respectively, imply that

$$w'(x) = -\frac{u_x(w(x),x)}{u_c(w(x),x)} + f_y(x,m(x))m'(x). \tag{3.12}$$

Equation (3.12) illustrates the change in the equilibrium wage for a worker who acquires a better job at the margin. The first term is the (negative) marginal rate of substitution of the job characteristic for consumption in a stable matching equilibrium. This represents compensating wage differentials, and is negative as predicted by the theory of equalizing differences. The second term on the right-hand side represents the marginal change in the output produced by the worker. It is positive when stable job matching is positively assortative (i.e., $m'(x) > 0$). Positively assortative matching induces a positive correlation between the two terms. Depending on which term is bigger, the equilibrium wage and the job characteristic may move in the same or opposite direction.

More importantly, the slope of the wage function $w'(x)$ is positively correlated with the slope of the job matching function $m'(x)$, given the positively assortative equilibrium job matching. If the distribution of worker productivity is widespread, under a postively assortative matching, it will be reflected in a very steep equilibrium matching function. In this case, the increase in output achieved by hiring a slightly better worker in terms of his/her productivity is significantly large at the margin. This leads to fierce competition among firms for workers with higher productivity, driving up equilibrium wages for better workers a great deal. It is less likely to observe compensating wage differentials and more likely to see patterns opposite to compensating differentials in sectors where matching is assortative and the matching function is increasing and very steep.

On the other hand, we can also summarize cases in which compensating wage differentials are more likely to be observed. First of all, according to Theorem 3.1, the observation of compensating wage differentials is more likely if any systematic matching pattern between the worker's productivity and the firm's job characteristic is not expected. Secondly, suppose that equilibrium job matching is positively assortative so that the equilibrium wage follows equation (3.12). If the distribution of worker productivity is concentrated so that the slope of the matching function is very flat (i.e., $m'(x)$ is positive but low), then it is more likely that compensating wage differentials will be observed even with a positively assortative equilibrium job matching. In fact, the case of $m'(x) = 0$ for all x is equivalent to the case without productivity heterogeneity. In this case wage differentials are comprised compensating wage differentials only.

Equation (3.12) also shows how to obtain the effect of compensating wage differentials even when the slope of the wage function is positive. If we regress the slope of the wage function on the slope of the matching function, then the intercept term should be negative and will represent the effect of compensating wage differentials on the wage.

Although the first-order condition (3.12) is derived from a model without heterogeneity in worker preferences, the model can easily incorporate this heterogeneity. Assume that there are N groups of workers (e.g., a group of males and a group of females, with $N = 2$ if males and females have different preferences over consumption and job characteristics). Let the preferences for workers in group i be represented by the utility function $u^i(c,x)$. This leads to N implicit competitive labor markets, one for each group of workers. Denoting the market wage function and the market matching function in the implicit market for group i by $w_i(x)$ and $m_i(x)$, respectively, the first-order condition (3.12) for the implicit market for group i is given by

$$w_i'(x) = f_y(x, m_i(x))m_i'(x) - \frac{u_x^i(w(x),x)}{u_c^i(w(x),x)} \text{ for } i = 1, \ldots, N.$$

Finally, in equilibrium, firm x is indifferent to which group it hires a worker from. This condition implies that for all x,

$$f(x, m_i(x)) - w_i(x) = f(x, m_j(x)) - w_j(x) \text{ for all } i, j.$$

4 Empirical Evidence

4.1 Estimation Strategy

In this section we propose an estimation method which identifies compensating wage differentials and apply it to the U.S labor market using the U.S. 2000 Census and job amenity data from the 4th edition of the Dictionary of Occupational Titles (DOT). Our approach takes advantage of the prediction of the theory summarized in Equation

(3.12). First we estimate matching and wage functions for each U.S. state in order to obtain the slopes of the matching function $m'(x)$ and the wage function $w'(x)$ in each state. Second, we regress the estimated wage function slopes $w'(x)$ on the estimated matching function slopes $m'(x)$ to generate an estimate of the compensating wage differential term in Equation (3.12).

We specify the empirical matching function as

$$\tilde{y} \quad = \quad \alpha_0 + \alpha_1 x + \alpha_2^{'} z + u, \tag{4.1}$$

where $\tilde{y}$ is years of education, acting as a proxy for worker skills, x is a job amenity index dervied from the DOT, and z is a vector of worker characteristics including experience (up to the quartic term) and dummy variables for female, nonwhite, living in Metropolitan Statistical Area (MSA), and married. The last term u is an error term which is independently distributed with zero mean. The coefficient α_1 corresponds to the slope of the matching function $m'(x)$. We use years of education as a proxy for worker skills, because it appears to be the best skill measurement available in the census.[7] The observed worker characteristics z are included in order to account for possible heterogeneous preferences. As discussed at the end of the last section, the existence of heterogeneous preferences does not affect the main prediction of the theory.

The empirical wage function is given by

$$\ln w \quad = \quad \beta_0 + \beta_1 x + \beta_2 \tilde{y} + \beta_3^{'} z + \nu, \tag{4.2}$$

where ν is independently distributed with mean zero. Notice that ν does *not* include an unobserved component of worker skills y, because matching of x and y is one-to-one; all components of y are fully absorbed in x through the matching process. To account for potential heterogeneous preferences and productivity not captured by y, we include the observed characteristics z along with years of education. For example, in addition to the skills y that are complementary with job characteristics in the production function, workers may possess skills s that contribute to productivity independent of x and y. In this case, the production function can be written as $f(x,y) + g(s)$. In a competitive market, the part of the output $g(s)$ that is generated by the skills s will be given

[7] If available, test scores, such as the AFQT scores found in the National Longitudinal Survey of Youth, may be a better proxy.

to workers as part of the their compensation regardless of which firm hires the worker, and hence such skills do not affect the matching mechanism: workers and jobs are not matched on the basis of s. The effects of s on the wage is controlled for by including $\tilde{y}$ and z in the wage function.

We estimate Equations (4.1) and (4.2) by OLS for each state and Washington D.C. to obtain 51 sets of estimates for α_1 and β_1. The estimate of the wage function slope β_1 is then regressed on that of the matching function slope α_1. Specifically, we estimate the empirical counterpart of Equation (3.12), which is specified as

$$b_s = \gamma_0 + \gamma_1 a_s + \varepsilon_s,$$

where a_s and b_s are estimates of α_1 and β_1 for state s, respectively, and ε_s is an error term with a zero mean. Remember that α_1 and β_1 correspond to $m'(x)$ and $w'(x)$ in Equation (3.12), respectively. Moreover, given Equation (3.12), we interpret γ_0 as the mean of compensating wage differential terms for consumers across states and γ_1 as the mean of marginal products of labor across states. When preferences and technology are heterogeneous across states, the compensating wage differential term and marginal product of labor also vary by state. Their deviations from the mean are collected by the error term ε_s. If between-state heterogeneity in preferences and technology is correlated with the matching function slope a_s, the parameter estimates for γ_0 and γ_1 are biased. However, the theory does not predict that the matching function is affected by preferences or technology. Instead, it predicts that matching is determined by the distributions of job characteristics and worker productivity alone. Although this argument does by no means prove consistency of the OLS estimates, the proposed method is internally consistent. We calculate the standard errors for γ_0 and γ_1 by a nonparametric bootstrap procedure in order to account for uncertainty about the estimates a_s and b_s obtained in the estimation's first step.

The key sources of identification in the proposed approach are the exogenous variation of worker and job distributions in each state and the theoretical prediction contained in Equation (3.12). One might consider an instrumental variable approach which identifies compensating wage differentials using the same exogenous variation. Namely, one may want to estimate the wage equation (4.2) using an instrument for the job amenity index x such as the state dummy variables or some other variables that vary across states. However, for the state dummy variables to be valid instruments, the wage function must be same across states or the state dummies must not have direct effects on wages. This exclusion restriction is not satisfied when preferences and/or technology differ across states. To the contrary, in the proposed approach the compensating wage differential is consistently estimated even when preferences and technology are heterogeneous across states. This is not to say that the proposed approach is superior to an instrumental variable approach; they require different assumptions. Nevertheless, we consider the assumptions required for the proposed approach to be more plausible for this specific empirical application.

4.2 Data

A sample of civilian male and female non self-employed workers in the non-agricultural sector between 18 and 65 years old are taken from the U.S. 2000 Census.[8] The large sample size of the census allows us to estimate the matching and wage functions for each state precisely, which is essential to the proposed estimation method. The job amenity variables are drawn from the 4th edition of the DOT. Specifically, we take seven variables related to the work environment that indicate whether a worker works in a job with the following characteristics: outdoor, extreme cold, extreme heat, wet, noise, hazard, and atmosphere. These DOT variables are merged to the census data, using the 1950 Census 3-digit occupation code provided by IPUMS and occupational cross walks. We then apply principal component analysis to construct a single job amenity index using the seven DOT variables. The first principal component serves as our measure of job amenity included in the estimation. The results for the principal component analysis are reported in Table 1. The factor loadings for the first principal component are all negative, meaning that the job amenity index decreases with these nasty work conditions. The first principal component explains as much as 57% of the variation in these variables To facilitate interpretation, the constructed job amenity index is normalized so that its mean is zero and the standard deviation is one at the national level.

Table 2 shows the means of the variables used in the regression models for six selected states. They are California, Florida, Illinois, Kansas, New York, and Texas.

[8] We use IPUMS-USA compiled by Ruggles, Alexander, Genadek, Goeken, and Schroeder (2010).

Table 1: Principal Component Analysis for Job Amenity Variables

	1st PC	2nd PC	3rd PC
Factor Loadings			
outdoor	−0.123	0.045	−0.112
exterme cold	−0.045	0.033	0.011
extreme heat	−0.085	0.082	0.049
wetness	−0.112	0.068	0.025
noises	−0.658	−0.747	0.080
hazards	−0.651	0.623	0.385
bad atmosphere	−0.326	0.198	−0.911
Proportion of Variance			
	0.566	0.209	0.100

Source: U.S. 2000 Census and the 4th edition of Dictionary of Occupational Titles
Note: The variables listed in the table are dummy variables indicating whether the job has work conditions characterized by outdoor, extremes of heat, cold, wetness, noises, hazards, and (bad) atmosphere and take one if exists and zero otherwise. The factor loadings and proportions of the variance explained are reported for up to the third principal component.

They are chosen for their geographical and labor market diversity.[9]

[9]The results for other states are available on request.

Table 2: Descriptive Statistics and Regression Results by State

	California	Florida	Illinois	Kansas	New York	Texas	USA
Education	12.926	13.031	13.182	13.186	13.387	12.673	13.090
Wage	18.783	15.754	17.502	14.681	18.984	15.854	16.667
Job Amenity	0.079	0.077	0.000	−0.061	0.074	−0.016	0.000
Experience	18.968	20.165	19.381	19.372	19.657	19.177	19.437
Female	0.477	0.508	0.492	0.504	0.503	0.483	0.496
Black	0.057	0.144	0.106	0.043	0.121	0.103	0.104
Married	0.539	0.549	0.584	0.624	0.549	0.603	0.577
Living in MSA	0.964	0.929	0.744	0.427	0.854	0.779	0.725
Slope of matching fn: q	1.204	0.810	0.902	0.791	0.946	1.080	0.888
Slope of wage fn: β	0.050	0.042	0.004	0.006	0.027	0.036	0.021
No. of obs.	654115	298510	266813	54413	383230	413272	5749281

Source: U.S. 2000 Census and the 4th edition of Dictionary of Occupational Titles. The standard errors for regression coefficients are in parenthesis. The explanatory variable in the matching equation is Job Amenity. Those in the wage regression are Job Amenity, Education, Experience (up to the 4th order polynomial term) and the dummy variables for female, nonwhite, living in Metropolitan Statistical Area (MSA), and the married. The job amenity index is normalized so that the mean is zero and the standard deviation is one at the national level.

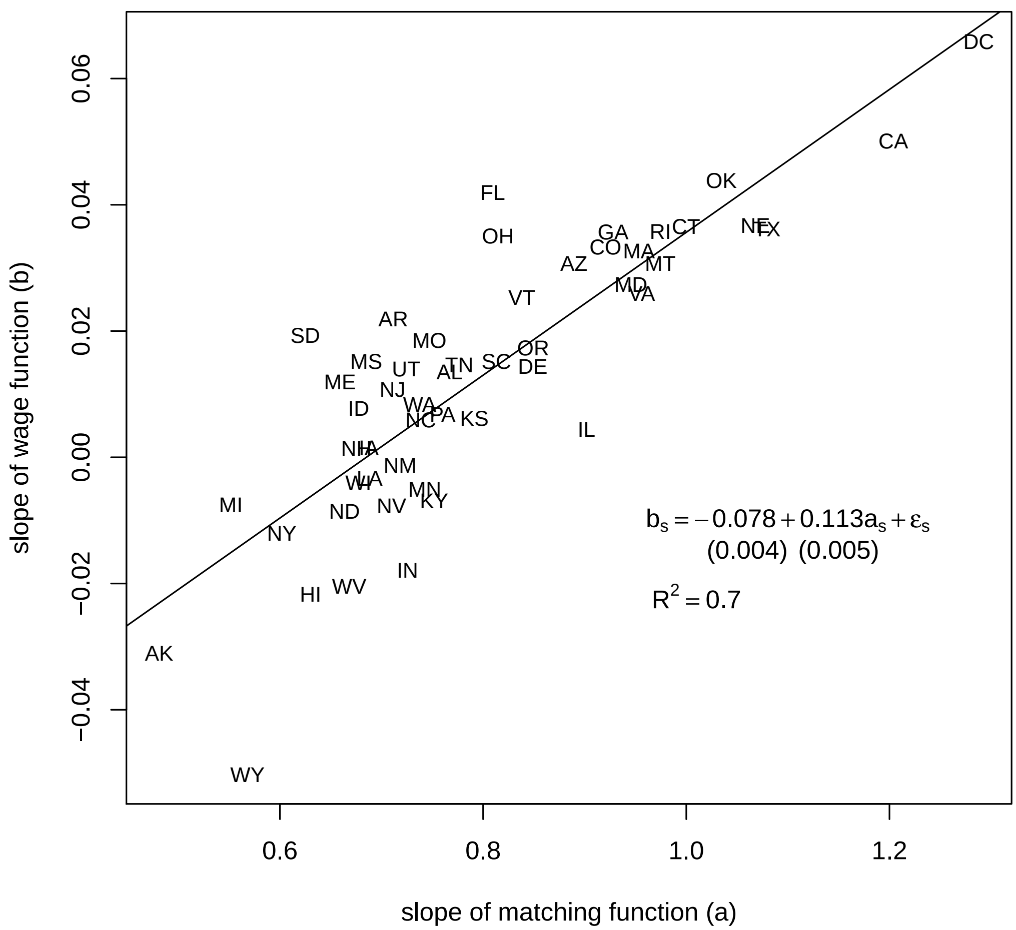
4.3 Estimation Results

We find worker-job matching is positively assortative in all 50 states and D.C. The results for the selected states are in Table 2. Among them, California, Texas, and New York have steeper sloped matching functions than Illinois, Florida, and Kansas, which implies that the worker skill gap for a given pair of occupations is greater in the former states. More specifically, when the job amenity index changes by one standard deviation, a worker's education increases by 1.2 years in California, while it increases by only 0.8 years in Kansas.

The estimation results for the wage equation (4.2) are also reported in Table 2. The estimated returns to job amenities are positive, instead of negative, in all the selected states. They are higher in California, Texas, and Florida than those in New York, Kansas, and Illinois. More specifically, an increase in the job characteristics by one standard deviation raises the wage by 5% in California, while in Kansas the wage rises by less than 1%. One might find these estimates counter-intuitive, because workers in a bad environment should be compensated by a higher wage. However, our theory suggests that the observed wage differences across occupations do not necessarily reflect compensating wage differentials. When workers and jobs are assortatively matched, the compensating wage differentials are confounded with the matching effect. Indeed, if the matching effect is strong enough, it exceeds the compensating wage differentials and the estimated coefficient for the job amenity index is positive. Notice that states with higher returns to the job characteristic also have steeper slopes in the matching functions, which is consistent with the prediction of the model as shown in Equation (3.12). This positive correlation between the slope of the matching function and the returns to the job characteristic can be seen not only for the selected states, but also for other states across U.S.

We plot in Figure 1 the slopes of the matching function and returns to job amenities for all 50 states and D.C using the estimation results for Equations (4.1) and (4.2). The returns to job amenities vary across states substantially. In the 5 states with the lowest returns, a one standard deviation increase of the job amenity index *decreases* the wage by 2-5%, while in the 5 states with the highest returns, the wage *increases* by 4-6%. This heterogeneity in the return to job amenities is largely explained by heterogeneity in the matching function, as can be seen in Figure 1. We regress the estimated returns

Figure 1: Slopes of Matching and Wage Functions



Note: The slopes for 50 U.S. states and Washington D.C. are plotted. The standard errors are in parenthesis and calculated by a nonparametric bootstrap with 300 replications.

to job amenities on the estimated coefficients of the matching function. The slope of the fitted line is 0.113 with a bootstrapped standard error of 0.005 and a coefficient of determination of 0.70. This result supports the theory's main empirical prediction that the returns to job characteristics are affected by the assortative matching of jobs and workers, and thus the observed wage differentials across jobs do not necessarily reflect compensating wage differentials.

Moreover, this second stage regression allows us to estimate compensating wage differentials. The compensating wage differential is obtained by having the slope of the matching function a approach zero, and is equal to the intercept of the second stage regression. This estimate implies that an increase in the job amenity index by one standard deviation *decreases* the wage by 8%, which is consistent with the concept of compensating wage differentials: workers are willing to accept a lower wage in exchange for a better job amenity. In contrast, the OLS estimates for the pooled U.S. data imply that an increase of the job amenity index by one standard deviation *increases* the wage by 2% as shown in Table 2. The OLS estimates for the compensating wage differential is positively biased by as much as 10 percentage points, which we find to be an economically substantial difference.

5 Conclusion

Rosen (1986) emphasizes that much more attention must be paid to the value of workers' productivity and the nature of sorting and selection along that dimension in order to facilitate a thorough understanding of compensating wage differentials; an important point given that productivity is arguably the most important worker characteristic. The main challenge this presents to analysis is the double transaction inherent in the labor market. The presence of a double transaction prevents us from applying known results that are suitable markets lacking a double transaction, such as the commodity market.

We explicitly show how compensating wage differentials and worker productivity differences affect the wage structure across jobs. The key insight of the theory is that the worker-job matching pattern drives the size of the worker productivity effect that appears in the wage. Our theory explains why the rate of return to job amenity varies across states and why it is often positive, contrary to the theory of equalizing differences. The empirical analysis shows that our theory is consistent with data from the U.S. Census and the DOT, and that the worker productivity effect greatly biases the OLS estimates of compensating wage differentials. Our matching theory is based on a continuum of workers and jobs andis applicable to a closed-form analysis under some parametric assumptions. In the appendix, we present our closed-form analysis with classes of Cobb-Douglas quasilinear payoff functions.

Our theory can be extended in several directions. We mention the two most important issues here. First of all, our theory is based on one dimensional productivity and job characteristics. It may be possible to show the existence of a stable matching pattern between workers with multidimensional productivity characteristics and jobs with multidimensional characteristics. However, it is critical to characterize the stable matching pattern because it enables us to uncover the intuition behind how each characteristic component is valued and how the valuation of each characteristic affects compensating wage differentials. Secondly, we study the competitive labor market without frictions because it highlights the matching pattern as the main driver preventing us from interpreting the wage differences between jobs as compensating wage differentials. However, workers and jobs may not be matched in a fully assortative way. It would be interesting to see how workers and jobs are matched in a frictional labor market and how job characteristics are valued in such an environment. We leave those problems for our future research.

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A Proofs

A.1 Proof of Lemma 3.1.

Let firm $\underline{x}$ be the firm whose job is the least attractive among those jobs available at firms that hire workers in equilibrium. Suppose that there is a firm that does not hire a worker in equilibrium and that its job characteristic $\acute{x}$ is higher than $\underline{x}$. The minimum wage that worker $m(\underline{x})$ is willing to accept from firm $\acute{x}$ is less than $w(\underline{x})$ because $\acute{x} > \underline{x}$. This minimum wage $\acute{w}$ is determined by $u(\acute{w},\acute{x}) = u(w(\underline{x}),\underline{x})$. When firm $\acute{x}$ offers wage $\acute{w}$, its profit is

$$f(\acute{x},m(\underline{x})) - \acute{w} > f(\underline{x},m(\underline{x})) - w(\underline{x}) \geq 0.$$

The first equality holds because (i) $\acute{x} > \underline{x}$ and the production function is non-decreasing in the job characteristic and (ii) $\acute{w} < w(\underline{x})$. The second inequality holds because the equilibrium profit for any firm is non-negative. Therefore, the profit for firm $\acute{x}$ is strictly positive, i.e., it is not a stable matching equilibrium if there is a firm whose job characteristic is higher than the job characteristic of any firm who hires a worker. Therefore, we have $T - F(\underline{x}) = 1$.

Consider the equilibrium profit for firm $\underline{x}$. Suppose that it is strictly positive: $f(\underline{x},m(\underline{x})) - w(\underline{x}) > 0$. Consider a firm x that does not hire a worker in equilibrium. It can hire worker $m(\underline{x})$ by offering the wage $\hat{w}(x)$ such that $u(\hat{w}(x),x) = u(w(\underline{x}),\underline{x})$. The profit for firm x by hiring worker $m(\underline{x})$ at wage $\hat{w}(x)$ is $f(x,m(\underline{x})) - \hat{w}(x)$. As $x \to \underline{x}$, $[f(x,m(\underline{x})) - \hat{w}(x)] \to [f(\underline{x},m(\underline{x})) - w(\underline{x})]$. Therefore, worker $m(\underline{x})$ and a firm whose job characteristic is close enough to $\underline{x}$ can be both better off by forming a match. This contradicts a stable matching equilibrium, so we must have $w(\underline{x}) = f(\underline{x}, m(\underline{x}))$. ■

A.2 Proof of Lemma3.2.

Consider any two workers, worker y and worker $\acute{y}$ such that $y > \acute{y}$. The equilibrium utility for worker y is $u(w(\tilde{x}(y)), \tilde{x}(y))$. The maximum wage that firm $\tilde{x}(\acute{y})$ is willing to offer to worker y is $\acute{w} = f(\tilde{x}(\acute{y}), y) - f(\tilde{x}(\acute{y}), \acute{y}) + w(\tilde{x}(\acute{y}))$. In a stable matching equilibrium, we have

$$u(w(\tilde{x}(y)), \tilde{x}(y)) \geq u(\acute{w}, \tilde{x}(\acute{y})). \tag{A.1}$$

Because $y > \acute{y}$, $f(\tilde{x}(\acute{y}), y) - f(\tilde{x}(\acute{y}), \acute{y}) > 0$. Therefore, we have $\acute{w} > w(\tilde{x}(\acute{y}))$. It means that

$$u(\acute{w}, \tilde{x}(\acute{y})) > u(w(\tilde{x}(\acute{y})), \tilde{x}(\acute{y})). \tag{A.2}$$

From (A.1) and (A.2), $u(w(\tilde{x}(y)), \tilde{x}(y)) > u(w(\tilde{x}(\acute{y})), \tilde{x}(\acute{y}))$. It shows that the worker's equilibrium utility is increasing in his/her characteristic. ■

A.3 Proof of Theorem 3.1.

For any arbitrary matching function $m(\cdot)$, consider any two matches in equilibrium: (i) one between firm x and worker $m(x)$; and (ii) the other between firm $\acute{x}$ and worker $m(\acute{x})$. It is necessary to have the following equality in equilibrium:

$$f(m(x)) - f(m(\acute{x})) = w(x) + \phi(x) - [w(\acute{x}) + \phi(\acute{x})], \tag{A.3}$$

if $x \neq \acute{x}$. Suppose not. Without loss of generality, consider the case

$$f(m(x)) - f(m(\acute{x})) < w(x) + \phi(x) - [w(\acute{x}) + \phi(\acute{x})].$$

Rearranging this inequality yields

$$f(m(x)) - w(x) < f(m(\acute{x})) - w(\acute{x}) - \phi(\acute{x}) + \phi(x). \tag{A.4}$$

The left-hand side of (A.4) is the equilibrium profit for firm x . Since firm x can hire worker $m(\acute{x})$ by offering wage w such that $w+\phi(x)$ is at least as high as $w(\acute{x})+\phi(\acute{x})$ (i.e., w is at least as high as $w(\acute{x})+\phi(\acute{x})-\phi(x)$), (A.4) shows that firm x and worker $m(\acute{x})$ can both be better off by forming a match at a wage higher than but arbitrarily close to $w(\acute{x})+\phi(\acute{x})-\phi(x)$. This contradicts that firm x hires worker $m(x)$ at wage $w(x)$ in equilibrium. Therefore, it is necessary for (A.3) to hold in equilibrium.

Consider the equilibrium profit $\pi(x) = f(m(x)) - w(m(x))$ for firm x . By (A.3), the equilibrium profit difference between firm x and firm $\acute{x}$ is

$$\pi(x)-\pi(\acute{x}) = \phi(x)-\phi(\acute{x}) \tag{A.5}$$

for any $x,\acute{x} \geq \underline{x}$. If $\acute{x}=\underline{x}$, $\pi(\underline{x})=0$ because of Lemma 3.1 and therefore (A.5) yields

$$\pi(x) = f(m(x)) - w(x) = \phi(x)-\phi(\underline{x})$$

for all $x \geq \underline{x}$. Therefore, the equilibrium wage becomes

$$w(x) = f(m(x)) - \phi(x) + \phi(\underline{x}) \tag{A.6}$$

for all $x \geq \underline{x}$.

Suppose that firms and workers match according to $\{m(\cdot), w(\cdot)\}$. Let us examine whether firm x has an incentive to hire any other worker, say worker $m(\acute{x})$. The minimum wage that can induce worker $m(\acute{x})$ to work for firm x is $w(\acute{x})+\phi(\acute{x})-\phi(x)$. Therefore, the profit for firm x associated with hiring worker $m(\acute{x})$ cannot be higher than $f(m(\acute{x}))-w(\acute{x})-\phi(\acute{x})+\phi(x)$. According to (A.3), this is exactly the same as the profit associated with hiring worker $y(x)$:

$$f(m(x)) - w(x) = f(m(\acute{x})) - w(\acute{x}) - \phi(\acute{x}) + \phi(x).$$

This shows that if (A.3) is satisfied, no firm has an incentive to hire any other worker if the firm can hire a worker following $\{m(\cdot), w(\cdot)\}$. Consider firm x who does not hire in equilibrium. Because its job characteristic x is lower than $\underline{x}$, it should offer a wage higher than the one firm $\underline{x}$ would offer to any worker if it wants to hire the worker. Because firm $\underline{x}$ cannot get a positive profit by hiring any worker, any firm x with $x < \underline{x}$

does not have an incentive to hire any worker.

Finally, let us examine whether a worker has an incentive to work for any other firm. If worker y wants to work for firm $x \geq \underline{x}$ other than firm $\tilde{x}(y)$, the maximum wage that firm x is willing to pay is $w = f(y) - f(m(x)) + w(x)$. By applying (A.3), we can see that the worker's utility from working for firm x cannot be higher than

$$f(y) - f(m(x)) + w(x) + \phi(x) = w(\tilde{x}(y)) + \phi(\tilde{x}(y)).$$

The right-hand side of the equation above is the utility that worker y receives from working for firm $\tilde{x}(y)$ at the market wage $w(\tilde{x}(y))$. Therefore, worker y cannot be better off by working for any other firm x such that $x \geq \underline{x}$. Suppose that worker y wants to work for firm x such that $x < \underline{x}$. The maximum wage that firm x is willing to pay is $w = f(y)$. Therefore, the worker's utility from working for firm x cannot be higher than $f(y) + \phi(x)$, which is strictly less than $f(y) + \phi(\underline{x})$ because $x < \underline{x}$. Furthermore, $f(y) + \phi(\underline{x})$ is exactly the same utility $w(\tilde{x}(y)) + \phi(\tilde{x}(y))$ that the worker receives from working for firm $\tilde{x}(y)$ because $w(\tilde{x}(y)) + \phi(\tilde{x}(y)) = f(y) + \phi(\underline{x})$ according to (A.6). Therefore, the worker does not have an incentive to work for any firm that does not hire any worker given $\{m(\cdot), w(\cdot)\}$. ■

B Additional Results (Not for Publication)

B.1 Closed Form Solution 1: Cobb-Douglas production function and non-separable utility function

We study the stable matching equilibrium through closed-form solutions. For the closed-form analysis on the effect of correlation on wage decomposition, we consider the class of Cobb-Douglas functional forms. These functional forms are general enough for the parameters to have various economic interpretations in the empirical/theoretical analysis. Let $f(x,y) = \gamma x^a y^b$ denote the production function, where $\gamma > 0$, $a \geq 0$ and $b > 0$. The parameters, a and b, determine the marginal rate of technical substitution between the job characteristic and the worker's productivity characteristic. The parameter γ may represent the technology across different industries at any given time or the time-varying technology in an industry. If $a > 0$, the job characteristic hasa common value and is also strictly complementary to the worker's productivity characteristic. If $a = 0$ in the production function, the job characteristic has only a private value.

Let $u(c,x) = \beta c^t x^s$ denote the worker's utility function, where $\beta > 0$, $s > 0$, and $0 < t \leq 1$. The worker's utility function is concave in consumption ($0 < t \leq 1$) and the job characteristic is strictly complementary to consumption. The parameters, t and s, determine the worker's marginal rate of substitution of the job characteristic for consumption. As s is increasing relative to t, the worker is willing to pay more (receive a lower wage) for a better job characteristic.

Homogenous labor productivity

First of all, consider the case where workers are homogenous in their labor productivity. Let $\hat{y}$ denote every worker's productivity. Let $\underline{x}$ be the lowest job characteristic from among the characteristics of the jobs that are filled in equilibrium. As discussed in section 2, firm $\underline{x}$ has zero profit in equilibrium, otherwise those firms whose job characteristics are lower than but arbitrarily close to $\underline{x}$ would make wage offers to the worker to attract him/her. Therefore, $w(\underline{x}) = f(\underline{x}, \hat{y}) = \gamma \underline{x}^a \hat{y}^b$. Because every worker has the same productivity, he/she will receive the same utility in equilibrium. Hence the equilibrium utility for the worker who is hired by firm x is

$$u(w(x),x) = \beta w(x)^t x^s = \beta \gamma^t \underline{x}^{at+s} \hat{y}^{bt} = u(w(\underline{x}),\underline{x}).$$

By taking the log transformation, the equilibrium wage is expressed by

$$\ln w(x) = \ln \gamma + \left(a + \frac{s}{t}\right) \ln \underline{x} + b \ln \hat{y} - \frac{s}{t} \ln x. \tag{B.1}$$

Wage equation (B.1) shows that the coefficients of the job characteristic are negative: a less preferred job is compensated by a higher wage and the worker is willing to accept a lower wage for a better job. This equation illustrates compensating wage differentials without heterogeneity in workerproductivity.

Heterogeneous labor productivity

Now consider the case where workers are heterogeneous in their productivity. Equation (3.12) leads us to a closed-form solution for the equilibrium wage equation when equilibrium matching is positively assortative. We assume that $\beta > 0$, $s > 0$, and $0 < t \leq 1$ for the worker's utility function $u(c,x) = \beta c^t x^s$. Because the job characteristic is strictly complementary to consumption in the worker's utility, stable job matching is positively assortative with or without strict complementarity between the job characteristic and the worker's productivity characteristic. This is due to condition (b) in Theorem 3.2. Assume that F and G are the probability distributions for the job characteristics and the worker's productivity characteristics, respectively. Theorem B.1 below fully characterizes the stable matching equilibrium.

Theorem B.1 *Given $u(c,x) = \beta c^t x^s$ and $f(x,y) = \gamma x^a y^b$, $\{m(\cdot), w(\cdot)\}$ is the stable matching equilibrium if (and only if) for all:*

1. $x \geq \underline{x}$, $1 - F(x) = 1 - G(m(x))$ *and*

2. $x \geq \underline{x}$,

$$w'(x) = b\gamma x^a m(x)^{b-1} m'(x) - \frac{sw(x)}{tx} \tag{B.2}$$

with the initial condition $w(\underline{x}) = \gamma \underline{x}^a m(\underline{x})^b$.

Proof of Theorem B.1. The assortative matching condition comes from Lemma 3.1 and condition (b) in Theorem 3.2. Consider the firm's problem (F1). Because the constraint is always binding, $w = w(\tilde{x}(y))\tilde{x}(y)^{\frac{s}{t}}x^{-\frac{s}{t}}$ is the wage that firm x would offer to worker y if it wants to hire the worker. Therefore, the profit associated with hiring worker y is

$$\gamma x^a y^b - w(\tilde{x}(y))\tilde{x}(y)^{\frac{s}{t}}x^{-\frac{s}{t}}.$$

Its first-order derivative with respect to y is

$$b\gamma x^a y^{b-1} - w'(\tilde{x}(y))\tilde{x}'(y)\tilde{x}(y)^{\frac{s}{t}}x^{-\frac{s}{t}} - \frac{s}{t}w(\tilde{x}(y))\tilde{x}'(y)\tilde{x}(y)^{\frac{s}{t}-1}x^{-\frac{s}{t}}. \tag{B.3}$$

If y is optimal for firm x (i.e., $\tilde{x}(y) = x$), then (B.3) is equal to zero at y and it is reduced to

$$b\gamma x^a y^{b-1} m'(x) - w'(x) - \frac{sw(x)}{tx} = 0. \tag{B.4}$$

Suppose that firm x wants to hire worker $\acute{y}$. Because (B.4) holds for firm $\tilde{x}(\acute{y})$ who hires worker $\acute{y}$, and $\tilde{x}(\cdot)$ is increasing in the worker's productivity characteristic, the first-order condition for firm x associated with hiring worker $\acute{y}$ becomes

$$b\gamma x^a \acute{y}^{b-1} - w'(\tilde{x}(\acute{y}))\tilde{x}'(\acute{y})\tilde{x}(\acute{y})^{\frac{s}{t}}x^{-\frac{s}{t}} - \frac{s}{t}w(\tilde{x}(\acute{y}))\tilde{x}'(\acute{y})\tilde{x}(\acute{y})^{\frac{s}{t}-1}x^{-\frac{s}{t}} \gtreqless 0,$$

if $\acute{y} \lesseqgtr y$. Therefore, it is optimal for firm x to hire worker y such that $\tilde{x}(y) = x$ at wage $w(x)$.

Consider the worker's problem (W1). Because the constraint is always binding, the wage that worker y is willing to accept is $w = \gamma x^a y^b - \gamma x^a m(x)^b + w(x)$ when he/she wants to work for firm x and the utility for the worker is equal to

$$\beta(\gamma x^a y^b - \gamma x^a m(x)^b + w(x))^t x^s.$$

Its first-order derivative with respect to x is

$$[t\beta(a\gamma x^{a-1}y^b - a\gamma x^{a-1}m(x)^b - b\gamma m(x)^{b-1}m'(x) + w'(x))\times \\ \left(\gamma y^b - \gamma m(x)^b + w(x)\right)^{t-1}x^s] + s\beta\left(\gamma y^b - \gamma m(x)^b + w(x)\right)^t x^{s-1}.$$

If x is optimal for the worker with y (i.e., $m(x) = y$), then the first-order derivative is equal to zero at x and it is reduced to

$$-b\gamma m(x)^{b-1}m'(x)+w'(x)+\frac{sw(x)}{tx}=0. \tag{B.5}$$

Suppose that worker y wants to work for firm $\acute{x}$. Because (B.5) holds for the worker who works for firm $\acute{x}$ and $m(\cdot)$ is increasing in the job characteristic, the first-order condition for worker y associated with working for firm $\acute{x}$ becomes

$$[t\beta(a\gamma\acute{x}^{a-1}y^b-a\gamma\acute{x}^{a-1}m(\acute{x})^b-b\gamma m(\acute{x})^{b-1}m'(\acute{x})+w'(\acute{x}))\times \left(\gamma y^b-\gamma m(\acute{x})^b+w(\acute{x})\right)^{t-1}\acute{x}^s]+s\beta\left(\gamma y^b-\gamma m(\acute{x})^b+w(\acute{x})\right)^t\acute{x}^{s-1}\gtrless 0.$$

if $\acute{x}\lesseqgtr x$. Therefore, it is optimal for worker y to work for firm x such that $m(x)=y$ at wage $w(x)$. Both (B.4) and (B.5) are identical to (B.2). Also, note that the wage for worker $m(\underline{x})$ is $w(\underline{x})=\gamma m(\underline{x})^b$ because of Lemma 3.1. ■

Positively assortative matching means that the firm whose job characteristic is ranked in the top $p\%$ hires the worker whose productivity is also ranked in the top $p\%$. Therefore, the market matching function $m(\cdot)$ is uniquely determined by $1-F(x)=1-G(m(x))$ for all $x\geq\underline{x}$. Given the increasing market matching function, (B.2) is the first-order condition for both problems (W1) and (F1). When this condition holds for every $x\geq\underline{x}$, it becomes the necessary and sufficient condition for the market wage function along with the increasing property of the market matching function. Finally, because firm $\underline{x}$ has the least attractive job among firms who hire workers, it obtains zero profit by giving its worker the total output as his/her wage.

For the closed-form solution for differential equation (B.2) with the initial condition $w(\underline{x})$, it is necessary to have a functional form of $m(x)$ that satisfies $1-F(x)=1-G(m(x))$. We derive the closed-form solution given the following functional form:

$$m(x)=kx^q, \tag{B.6}$$

where $k>0$ and $q>0$. k is the "shift" parameter and q is the "relative spacing" parameter. Given k, the relative spacing parameter q shows the relative heterogeneity of the worker's productivity characteristic to the job characteristic. The validity of the functional form in (B.6) depends how well it approximates the actual distributions of the job characteristic and the worker's productivity characteristic. Nonetheless, (B.6) is sufficiently rich to allow for the in-depth closed-form analysis and its implications. This functional form can be derived under several reasonable distributions for the job characteristic and the worker's productivity characteristic.[10] Given (B.6), the first-order condition in (B.2) is

$$w'(x) = qb\gamma k^b x^{a+qb-1} - \frac{sw(x)}{tx}.$$

If we assume that $\underline{x}$ is arbitrarily close to zero, the solution to this differential equation becomes

$$w(x) = \frac{bq\gamma k^b}{a+qb+(s/t)} x^{a+qb}. \tag{B.7}$$

Corollary B.1 *Taking the log transformation of (B.7), the equilibrium wage equation is*

$$\ln w(x) = \left[\ln\left(\frac{b\gamma q}{a+qb+(s/t)}\right) + b\ln k\right] + (a+qb)\ln x \tag{B.8}$$

for all $x > \underline{x}$.

Surprisingly, the coefficient of the log wage is positive instead of negative. This is because the coefficient of the log wage shows the overall net effect of the worker's productivity and the compensating wage differentials on the equilibrium wage, and the effect of the worker's productivity is bigger than the effect of the compensating wage differentials. To clarify this point, note that the equilibrium wage change at the margin expressed in the first-order condition is

$$w'(x) = bq\gamma k^b x^{a+qb-1} - \left(\frac{(s/t)}{a+qb+(s/t)}\right) bq\gamma k^b x^{a+qb-1}. \tag{B.9}$$

The first term on the right-hand side is the marginal increase in the equilibrium

[10] The assortative matching function takes the form specified in (B.6) under the several classes of well known distributions. For example, the assortative matching function follows (B.6) when the distributions of the job characteristics and the worker's productivity charactyeristic follows the class of Weibull distributions, the class of Pareto distributions, the class of Fréchet distributions, the class of Gumbel distributions, or the class of log-normal distributions. The Pareto distributions are very helpful in approximating the distributions of many economic variables such as individual income levels, city sizes, insurance claims, and standardized price returns on individual stocks among many others. It quite nicely approximates firm size and also possibly the CEO's skill in the matching market for CEOs (Gabaix and Landier 2008).

wage with respect to the worker's productivity. It is expressed in terms of the job characteristic because of positively assortative matching, and it is positive. The second term is the marginal decrease in the equilibrium wage due to compensating wage differentials. Because $\left(\frac{(s/t)}{a+qb+(s/t)}\right) < 1$, the marginal effect of the worker's productivity on the equilibrium wage is always greater than the marginal effect of the compensating wage differentials. Therefore, the equilibrium wage is increasing in the job characteristic: $w'(x) > 0$. This is why the coefficient of the log wage is positive instead of negative.

As the (relative) heterogeneity of worker productivity increases (i.e., q increases), the first term on the right hand in (B.9) (marginal wage increase due to the worker's productivity) increases at a faster rate than the decrease in the second term (marginal wage decrease due to compensating wage differentials). Therefore, the rate of increase in the equilibrium wage with respect to the job characteristic increases in q. This is why the coefficient of the job characteristic in the equilibrium wage equation in (B.8) increases in q.

We can also compare the wage equation (B.8) with heterogeneity on the worker's productivity to (B.1) without it. In fact, the wage equation without heterogeneity in worker productivity can be viewed as the case with $q = 0$ in the matching function $m(x) = kx^q$. Therefore, the worker's productivity in the homogenous case becomes $\hat{y} = m(x) = kx^0 = k$ for all x. From (B.1) and (B.8), one can see that the coefficient of $\ln x$ is negative (i.e., $-s/t$) for $q = 0$. However, it is positive for any positive value of q and it converges to a as $q \to 0$. Therefore, the closed-form solution for the wage equation is discontinuous at $q = 0$ while the first-order condition (3.12) for the equilibrium wage equation does not necessarily imply a discontinuous solution.

B.2 Closed Form Solution 2: Cobb-Douglas production function and quasilinear utility function

Positively assortative matching can arise in equilibrium even with a quasilinear utility function if the job characteristic has a common value. Suppose that the worker's utility function is $u(c,x) = c + \beta x^s$ with $\beta > 0$ and $s > 0$ and the production function follows the general class of Cobb-Douglas functions, $f(x,y) = \gamma x^a y^b$ with $\gamma > 0$, $a > 0$, and $b > 0$.

Homogenous labor productivity

First consider the case where workers are homogenous in their productivity. Again, letting $\hat{y}$ denote every worker's productivity characteristic, every worker receives the same utility level. Because $u(w(\underline{x}),\underline{x}) = \gamma\underline{x}^a\hat{y}^b + \beta\underline{x}^s$ and every worker receives the same utility, we have $w(x) + \beta x^s = \gamma\underline{x}^a\hat{y}^b + \beta\underline{x}^s$, which implies that the equilibrium wage is

$$w(x) = \gamma\underline{x}^a\hat{y}^b + \beta\underline{x}^s - \beta x^s. \tag{B.10}$$

Heterogeneous labor productivity

Now consider the case where workers are heterogeneous in their productivity. While the marginal utility of consumption and the marginal utility of the job characteristic are independent of each other, the job characteristic is strictly complementary to the worker's productivity characteristic in the production function, so the marginal productivity of the worker is increasing in the job characteristic. By condition (a) in Theorem 3.2, this induces positively assortative stable matching between the worker's productivity characteristic and the job characteristic.

Theorem B.2 *Given $u(c,x) = c + \beta x^s$ and $f(x,y) = \gamma x^a y^b$ with $a > 0$, $\{m(\cdot), w(\cdot)\}$ is the stable matching equilibrium if (and only if) for all:*

1. $x \geq \underline{x}$, $1 - F(x) = 1 - G(m(x))$ *and*

2. $x \geq \underline{x}$,

$$w'(x) = b\gamma x^a m(x)^{b-1} m'(x) - s\beta x^{s-1} \tag{B.11}$$

with the initial condition $w(\underline{x}) = \gamma\underline{x}^a m(\underline{x})^b$.

Equation (B.11) is an expression of the first-order condition (3.12) for the firm's problem (F1) and the worker's problem (W1). Together with positively assortative job matching, it becomes the necessary and sufficient condition for the stable matching equilibrium.[11] Given the functional form $m(x) = kx^q$ for positively assortative matching, we can solve differential equation (B.11) for the equilibrium wage.

[11]Theorem B.2 can be proved similarly as the proof for Theorem B.1.

Corollary B.2 *The equilibrium wage equation is*

$$w(x) = \left(\gamma k^b \underline{x}^{a+bq} - \frac{b\gamma k^b q}{a+qb}\underline{x}^{a+qb} + \beta \underline{x}^s\right) + \left(\frac{b\gamma k^b q}{a+qb}\right) x^{a+qb} - \beta x^s \tag{B.12}$$

for all $x \geq \underline{x}$.

In the equilibrium wage equation (B.12), the second term on the right-hand side is the part of the wage that is attributed to the worker's productivity and the third term reflects the compensating wage differential. Because the worker's productivity characteristic is positively assortative with the job characteristic, the part of the wage that is attributed to the worker's productivity is also explained by the job characteristic as shown in the second term. While compensating wage differentials and the part of wage attributed to the worker's productivity are explained by the job characteristic, the separability of the utility function makes the part of the wage that is attributed to the worker's productivity separable from the compensating wage differentials in the equilibrium wage equation. The coefficient of the third term is negative as the theory of equalizing differences predicts. Separability of the utility function makes only the part of wage attributed to the worker's productivity depend on the heterogeneity of the worker's productivity and it increases as the heterogeneity of the worker's productivity increases. Finally, when the worker's utility function is quasilinear, the coefficient of $\ln x$ is always negative (i.e., $-\beta$), from the wage equations (B.10) and (B.12), regardless of the heterogeneity of the worker's productivity. The wage equation (B.12) at $q = 0$ is not the same as the wage equation (B.10). Therefore, the solution for the wage equation is discontinuous at $q = 0$ in the case of the quasilinear utility function as it is in the case of the non-separable utility function.