Math camp: micro
Practice questions

1. Consider a simple national-income model

$$\begin{aligned} Y &= C + I_0 + G_0 \\ C &= \alpha + \beta Y \end{aligned}$$

where $\alpha > 0$ and $0 < \beta < 1$. Solve the system of equations by Cramer's rule. Derive the following matrix

$$J = \begin{bmatrix} \frac{\partial Y^*}{\partial \alpha} & \frac{\partial Y^*}{\partial \beta} \\ \frac{\partial C^*}{\partial \alpha} & \frac{\partial C^*}{\partial \beta} \end{bmatrix}$$

2. A firm has a production function $Q = f(K, L)$, where Q is the quantity of output, K is the amount of physical capital, L is the amount of labour. Suppose that this production function is an implicit function of $F(Q, K, L) = 0$. Derive the marginal rate of technical substitution from $F(Q, K, L) = 0$, that is,

$$\left.\frac{dK}{dL}\right|_{dQ=0}$$

3. The equilibrium value of the variable x is the solution of the equation

$$f(x, a, b) + g(x, k(a)) = 0$$

where a and b are exogenously given and f, g, and k are differentiable functions. How is the equilibrium value of x affected by a change in a (holding b constant)?

4. Consider a very simple national-income model (IS-LM) without taxation. The equilibrium in the goods market is described by the equation:

$$Y = C(Y) + I(r), \tag{1}$$

where $C(Y)$ specifies the aggregate consumption as a function of the national income Y, and $I(r)$ specifies the aggregate investment as a function of the interest rate r. The equilibrium in the money market can be described as the following equation

$$M_0^s = L(Y, r), \tag{2}$$

where M_0^s is the exogenously given money supply and $L(Y, r)$ specifies the money demand as a function of the national income and the interest rate. Let $Y^*(M_0^s)$ and $r^*(M_0^s)$ denote the equilibrium national income and the equilibrium interest rate respectively. Assume that all functions are differentiable. Derive $dY^*(M_0^s)/dM_0^s$ and $dr^*(M_0^s)/dM_0^s$, assuming that they exist.

5. Consider the following two sets:

$$\begin{aligned} A &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq 2 \text{ and } x_2 \leq 4\} \\ B &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0 \text{ and } x_2 \geq 2\} \end{aligned}$$

   Determine whether each of A, B, and $A \cap B$ is bound, closed, and compact.

6. Consider a twice-differentiable function $f : X \to \mathbb{R}$ with $X \subset \mathbb{R}^3$. Suppose that the Hessian matrices evaluated at two different x and x' are given by

$$H(x) = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & -1 \\ 2 & -1 & 0 \end{bmatrix} \text{ and } H(x') = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   Determine whether each of the two Hessians is PD, ND, PSD, NSD or indefinite.

7. Consider the function $f : X \to \mathbb{R}$ with $X = \mathbb{R}^2$ such that for any $(x_1, x_2) \in X$,

$$f(x_1, x_2) = -x_1^2 + x_1 x_2 - x_2^2 + 2x_1 + x_2$$

   Find all the local maximizers.

8. Consider a function $f : X \to \mathbb{R}$ with $X = \mathbb{R}^3$. The function f satisfies $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1 - x_3$ for all $(x_1, x_2, x_3) \in X$.

(a) Does a global minimum of f exist? If it does, derive it. If it does not exist, explain why it does not exist.

(b) Does a global maximum of f exist? If it does, derive it. If it does not exist, explain why it does not exist.

9. Suppose that $f(x)$ is a concave function and $h(x)$ is a convex function. Is $g(x) = f(x) - h(x)$ is a convex function or a concave function? Prove your answer precisely.

10. Consider a firm that produces two goods, good 1 and good 2 in a competitive market. The price of good i per unit is denoted by p_i for $i = 1, 2$. The firm takes the prices as given. The cost of producing x_1 units of good 1 and x_2 units of good 2 is given by $C(x_1, x_2) = x_1^2 + x_1 x_2 + 2x_2^2$. Find out the stationary point of the firm's profit function. Show that it maximizes the firm's profit.

11. Consider a monopolist who can sell his product in the two different markets. The inverse demand functions in the two markets are given by

$$\begin{aligned} p_1 &= 100 - q_1 \\ p_2 &= 120 - 2q_2 \end{aligned}$$

where p_i is the price of the product per unit and q_i is the quantity of the product demanded in market i $(i = 1, 2)$. The total cost of producing $q_1 + q_2$ units of the product is

$$c = 20(q_1 + q_2)$$

Write down the monopolist's profit when he sells q_1 units of the product in market 1 and q_2 units of the product in market 2. Solve the profit maximization problem and derive the profit-maximizing quantities, one for each market. Derive the prices of the good, one for each market, at the solution.

12. Consider a problem for local maximizer/minimizers. The obejctive function is $x^2 + y^2$ and the equality constraint is $4x^2 + 2y^2 = 4$. Find the four points that satisfies the first-order conditions for the Lagrangian function. Find local maximizers and local minimizers among them.

13. Consider the problem

$$\max_{x,y} x^a y^b \text{ subject to } px + y = m,$$

where $a > 0$, $b > 0$, $p > 0$, and $m > 0$, and the objective function $x^a y^b$ is defined on the set of all points (x, y) with $x \geq 0$ and $y \geq 0$. Find a solution if it exists.

14. The objective function is $f(x, y) = x - y^2$. The problem is to find (x, y) that maximizes $f(x, y)$ subject to (i) $x - y \leq 0$, (ii) $x \geq 0$ and (iii) $y \geq 0$. Set up the Lagrangean function. Find the solution to the problem if it exists.

15. Consider a maximization problem with inequality constraints. The objective function is given by $f(x_1, x_2) = x_1 + x_1 x_2$. The problem is to find (x_1^*, x_2^*) that maximizes $f(x_1, x_2)$ subject to (i) $x_1 + a x_2 \leq b$, (ii) $x_1 \geq 0$ and (iii) $x_2 \geq 0$. Assume that $a > 0$ and $b > 0$.

16. Prove that if function $g(x)$ is a concave function, then it is a quasiconcave function.

17. Consider the sequence of real numbers $\{x_k\}$ defined by

$$x_1 = \sqrt{2}, \qquad x_{k+1} = \sqrt{x_k + 2}, \;\; k = 1, 2, \ldots$$

Use Theorem 2.0.2 to prove that the sequence is convergent and find its limit. (Hint: Prove by induction that $x_k < 2$ for all k. Then prove that the sequence is (strictly) increasing).

18. Determine the $\overline{\lim}$ and $\underline{\lim}$ of the following sequences of real numbers

    (a) $\{x_k\} = \{(-1)^k\}$
    (b) $\{x_k\} = \left\{(-1)^k \left(2 + \frac{1}{k}\right) + 1\right\}$

19. Prove that the sequence of real numbers $\{x_k\}$ with the general terms $x_k = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2}$ is a Cauchy sequence.

20. Find all possible limits of subsequences of the sequence,

$$\{x_k\} = \left\{1 - \frac{1}{k} + (-1)^k\right\}$$

21. Show that if $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are points in $\mathbb{R}^n$, then $d(x, y) \leq \sum_{i=1}^n |x_i - y_i|$.

22. Show by an example that the union of infinitely many closed sets need not be closed. (Hint: Look at $\cup_{i=1}^\infty A_i$, where $A_i = \{1/i\}$ for $i = 1, 2, \ldots$)

23. Examine the convergence of the following sequence

    (a) $x_k = \left(\frac{1}{k}, 1 + \frac{1}{k}\right)$

    (b) $x_k = \left(1 + \frac{1}{k}, \left(1 + \frac{1}{k}\right)^k\right)$

    (c) $x_k = \left(\frac{k+2}{3k}, \frac{(-1)^k}{2k}\right)$

24. Give examples of subsets S of $\mathbb{R}$ and continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

    (a) S is closed, but $f(S)$ is not closed

    (b) S is open but $f(S)$ is not open

    (c) S is bounded, but $f(S)$ is not bounded.

25. Consider the function f defined for all x in $[0, 1]$ by $f(x) = \frac{1}{2}(x + 1)$. Prove that f maps $[0, 1]$ into itself and find a fixed point. Suppose that f is defined for all x in $(0, 1)$. Prove that f maps $(0, 1)$ into itself but f has no fixed point. Why does not Brouwer's fixed point theorem apply?