# Sets, Functions and Euclidean Space

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# Contents




**Abstract**

This note includes some basics for logic, sets, and the Euclidean space and it freely borrows from Sydsæster, et al. (2005) and Jehle and Reny (2000).

# Chapter 1

# Some Basics

## 1.1 Elements of Logic

Necessity and sufficiency are fundamental logical notations. Consider any two statements, $A$ and $B$.When we say, "$A$ is necessary for $B$," we mean that $A$ must hold or be true for $B$ to hold or be true. For $B$ to be true requires $A$ to be true, so whenever $B$ is true, we know that $A$ must also be true. So we might have said, instead, that "$A$ is true if $B$ is true," or simply that "$A$ is implied by $B$" ($B \Rightarrow A$)

Suppose we know that "$B \Rightarrow A$" is a true statement. What if $A$ is not true? Because $A$ is necessary for $B$, when $A$ is not true, then $B$ cannot be true either. So, not-$B$ is implied by not-$A$ ($\tilde{}A \Rightarrow \tilde{}B$). This form of the original statement is called the contrapositive form. Contraposition of the arguments in the statement reverses the direction of implication for the true statement.

The notion of necessity is distinct from that of sufficiency. When we say, "$A$ is sufficient for $B$," we mean that whenever $A$ holds, $B$ must hold. We can say, "$A$ is true only if $B$ is true," or that "$A$ implies $B$" ($A \Rightarrow B$). Once again, whenever the statement $A \Rightarrow B$ is true, the contrapositive statement, $\tilde{}B \Rightarrow \tilde{}A$ is also true.

Two implications, $A \Rightarrow B$ and $A \Leftarrow B$, can both be true. When this is so, we say that "$A$ is necessary and sufficient for $B$" or that "$A$ is true if and only if $B$ is true, or "$A$ is iff $B$." When $A$ is necessary and sufficient for $B$, we say that the statements $A$ and $B$ are equivalent and write "$A \Longleftrightarrow B$."

## 1.2 Theorems and Proofs

Important ideas in the economics literature are often stated in the form of mathematical theorems. Mathematical theorems usually have the form of an implication or an equivalence, where one or more statements are alleged to be related in particular ways. Suppose that we have the theorem "$A \Rightarrow B$." Here, $A$ is called the premise and $B$ the conclusion. To prove a theorem is to establish the validity of its conclusion given the truth of its premise and several methods can be used to do that.

1. Constructive proof (direct proof): We assume that $A$ is true, deduce various consequences of that and use them to show that $B$ must also hold.

2. Contrapositive proof: We assume that $B$ does not hold, then show that $A$ cannot hold. This approach takes advantage of the logical equivalence between the claims, $A \Rightarrow B$ and $\tilde{}B \Rightarrow \tilde{}A$, noted earlier, and essentially involves a constructive proof of the contrapositive to the original statement.

3. Proof by contradiction: The strategy is to assume that $A$ is true, assume that $B$ is not true, and attempt to derive a logical contradiction. This approach relies on the fact that if $A \Rightarrow \tilde{}B$ is false, then $A \Rightarrow B$ must be true. Sometimes, proofs by contradiction can get the job very efficiently yet because they involve no constructive chain of reasoning between $A$ and $B$ as the other two do, they seldom illuminate the relationship between the premise and the conclusion.

If we assert that $A$ is necessary and sufficient for $B$, or that $A \Longleftrightarrow B$, we must give a proof in both directions. That is, both $A \Rightarrow B$ and $B \Rightarrow A$ must be established before a complete proof of the assertion has been achieved. Finally, citing a hundred examples can never prove a certain property always holds, citing one solitary counterexample can disprove that the property always holds.

## 1.3 Sets

A set is a collection of objects. These objects are called the elements of the set. We use the notation $x \in S$ to indicate that $x$ is an element of $S$

(or belongs to $S$ or is a member of $S$). If $A$ and $B$ are two sets such that every element of $A$ is also an element of $B$, then $A$ is a subset of $B$ and one writes $A \subseteq B$ (read as "$A$ is a subset of $B$" or "$A$ is included in $B$" or "$B$ includes $A$") The set $A$ is a proper subset of $B$ if $A \subseteq B$ and $A \neq B$; sometimes one writes $A \subsetneq B$ in this case. The symbol $\subseteq$ is called the inclusion symbol. Some authors use $\subset$ as the inclusion symbol and some use $\subseteq$ for the inclusion and reserve $\subset$ for the proper inclusion. It is clear that $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$. It is also easy to see that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. The empty set, $\emptyset$, is a set with no elements at all and it is a subset of every set. The collection of all subsets of a set $A$ is also a set, called the power set of $A$ and denoted by $\mathcal{P}(A)$. Thus, $B \in \mathcal{P}(A) \Longleftrightarrow B \subseteq A$.

The order of the elements in a set specification such as $\{a, b, \cdots, t\}$ does not matter. Thus, in particular $\{a, b\} = \{b, a\}$. However, on many occasions one is interested in distinguishing between the first and the second elements of a pair. One such example is the coordinates of a point in the $xy$-plane. These coordinates are given as an ordered pair $(a, b)$ of real numbers. The important property of ordered pairs is that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. Once ordered pairs are available, ordered triples quadruples, etc. are defined by $(a, b, c) = ((a, b), c)$, $(a, b, c, d) = ((a, b, c), d)$, etc. Of course, there is a natural one-to-one correspondence $((a, b), c) \leftrightarrow (a, (b, c))$. The important thing again is that $(a, b, c) = (d, e, f)$ if and only if $a = d$, $b = e$, and $c = f$.

If $A$ and $B$ are sets, their Cartesian product is the set $A \times B$ consisting of all ordered pairs $(a, b)$ such that $a \in A$ and $b \in B$. Similarly, the Cartesian product of the sets $A, B$, and $C$ is the set of all ordered triples $(a, b, c)$ such that $a \in A$, $b \in B$, and $c \in C$. The natural one-to-one correspondence $((a, b), c) \leftrightarrow (a, (b, c))$ referred to above gives a one-to-one correspondence between $(A \times B) \times C \leftrightarrow A \times (B \times C)$, os one can well identify the two and write either product simply as $A \times B \times C$. The Euclidean plane $\mathbb{R}^2$ is the Cartesian product $\mathbb{R} \times \mathbb{R}$. More generally, $\mathbb{R}^m = \mathbb{R} \times \cdots \times \mathbb{R}$ ($m$ factors) and there is a natural identification between $\mathbb{R}^m \times \mathbb{R}^n$ and $\mathbb{R}^{m+n}$

There is often a need to go beyond ordered pairs. Suppose that for each $i$ in some set $I$, we specify an object $a_i$ (which can be a number, a set, or any other entity). Then these objects form an indexed set $\{a_i\}_{i \in I}$ with $I$ as its index set. In formal terms, an indexed set is a function whose domain is the index set. There is an important difference between the indexed set $\{a_i\}_{i \in I}$ and the set of all the values $a_i$. For example, an $n$-vector $x = (x_1, \cdots, x_n)$ is an indexed set with $\{1, 2, \cdots, n\}$ as its index set. Here the order of the elements does matter and multiple occurrences of the same value will also matter. Thus, the 5-dimensional vector $(3,-1,3,-1,-2)$ is different from the vector $(3,-2,-1,3,-1)$, whereas the sets $\{3,-1,3,-1,-2\}$ and $\{3,-2,-1,3,-1\}$ are equal. Indexed sets make it possible to talk about sets whose elements appear in some specific order and with possible repetitions. A sequence is an indexed set $\{a_k\}_{k\in\mathbb{N}}$ with the set $\mathbb{N}$ of natural numbers as its index set. One often writes $\{a_k\}_{k=1}^{\infty}$.

## 1.4 Relations

Any ordered pair $(s,t)$ associates an element $s \in S$ to an element $t \in T$. The sets $S$ and $T$ need not contain numbers; they can contain anything. Any collection of ordered pairs is said to constitute a binary relation between the sets $S$ and $T$.

A binary relation is defined by specifying some meaningful relationship that holds between the elements of the pair. For example, let $S$ be the set of cities {Washington, London, Paris, Ottawa} and $T$ be the set of countries {United States, England, Canada, Germany}. The state "is the capital of" then defines a relation between these two sets that contains the three elements: {(Washington, United States), (London, England), (Ottawa, Canada)}. As this example shows, a binary relation $R$ on $S \times T$ is always a subset of $S \times T$. When $s \in S$ bears the specified relationship to $t \in T$, we denote membership in the relation $R$ in one of two ways; Either we writes $(s,t) \in R$ or more commonly we simply write $sRt$. When a binary relation is a subset of the product of one set $S$ with itself, we say that it is a relation on the set $S$.

**Example 1.4.1** *Let $S$ be the closed unit interval $S = [0,1]$. Illustrate binary relation $\geq$ on $S$.*

We can build in more structure for a binary relation on some set by requiring that it possesses certain properties.

**Definition 1.4.1** *A relation $R$ on $X$ is reflexive if $xRx$ for all $x$ in $X$.*

**Definition 1.4.2** *A relation $R$ on $X$ is transitive if $xRy$ and $yRz$ implies $xRz$ for any three elements $x, y$, and $z$ in $X$.*

**Definition 1.4.3** *A relation $R$ on $X$ is complete if for all $x$ and $y$ in $X$, at least one of $xRy$ or $yRx$ holds.*

**Definition 1.4.4** *A relation $R$ on $X$ is symmetric if $xRy$ implies $yRx$.*

**Definition 1.4.5** *A relation $R$ on $X$ is anti-symmetric if $xRy$ and $yRx$ implies $x = y$.*

A partial ordering on $X$ is a relation on $X$ that is reflexive, transitive, and anti-symmetric. If a partial ordering is complete, it is called a linear (or total) ordering.

**Example 1.4.2** *Show that the relation $\leq$ on $\mathbb{R}$ is a linear ordering.*

## 1.5 Functions

A function is a very common though very special kind of relation. We say that the function $f$ is a mapping from $X$ to $Y$ and write $f : X \to Y$. Specifically, a function is a relation that associates each element of $X$ with a single, unique element of $Y$. Thus, $f$ is single-valued and operates on every $x$ in $X$. One usually writes $f(x) = y$ instead of $xfy$. For any $x$ in $X$, $f(x)$ is called the image of $x$ under $f$. The set $X$ is the domain and $Y$ is the range of $f$. The image set of $f$ includes images of all elements in $X$.[1]The last formulation is an example of a somewhat sloppy notation that is often used when the meaning is clear from the context. The graph of $f$ is the set

$$\text{graph}(f) = \{(x, y) \in X \times Y : y = f(x)\}$$

This is of course the same as the relation $f$ defined in the previous subsection.

If $f(x) = y$, one also writes $x \mapsto y$. The squaring function $s : R \to R$, for example, can then be written as $s : x \mapsto x^2$. Thus, $\mapsto$ indicates the effect of the function on an element of the domain. If $f : X \to Y$ is a function and $S \subset X$, the restriction of $f$ to $S$ is the function $f|_S$ defined by $f|_S(x) = f(x)$ for every $x \in S$. A function is said to be injective or one-to-one if $f(x) \neq f(x')$ whenever $x \neq x'$ . If the image set is equal to the range - if for every $y \in Y$, there is $x \in X$ such that $f(x) = y$, the function is said to be surjective or onto. If a function is one-to-one and onto, then an inverse function $f^{-1} : Y \to X$ exists that is also one-to-one and onto. The composition of a function $f : A \to B$ and a function $g : B \to C$ is the function $g \circ f : A \to C$ given by $(g \circ f)(a) = g(f(a))$ for all $a \in A$.

[1]Analysts are apt to use the word ”range” to denote what we have called the image set of $f$. In this case, $Y$ is called the codomain of $f$.

### 1.5.1 Least Upper Bound Principle

A set $X$ of real numbers is bounded above if there exists a real number $b$ such that $b \geq x$ for all $x$ in $X$. This number $b$ is called an upper bound for $S$. A set that is bounded above has many upper bounds. A least upper bound for the set $X$ is a number $b^*$ that is an upper bound for $S$ and is such that $b^* \leq b$ for every upper bound $b$. The existence of a least upper bound is a basic and non-trivial property of the real number system.

**Least Upper Bound Principle** Any non-empty set of real numbers that is bounded above has a least upper bound.

A set $S$ can have at most one least upper bound, because if $b_1^*$ and $b_2^*$ are both least upper bounds for $X$, then $b_1^* \leq b_2^*$ and $b_1^* \geq b_2^*$, and thus $b_1^* = b_2^*$. The least upper bound $b^*$ of $X$ is often called the supremum of $X$. We write $b^* = \sup X$ or $a^* = \sup_{x \in X} x$.

A set $X$ is bounded below if there exists a real number $a$ such that $x \geq a$ for all $x$ in $X$. The number $a$ is called a lower bound for $X$. A set $X$ that is bounded below has a greatest lower bound $a^*$ with the property $a^* \leq x$ for all $x$ in $X$ and $a^* \geq a$ for all lower bounds $a$. The number $a^*$ is called the infimum of $X$ and we write $a^* = \inf X$ or $b^* = \inf_{x \in X} x$. Thus,

sup $X$ = the least number greater than or equal to all numbers in $X$.

inf $X$ = the greatest number less than or equal to all numbers in $X$.

If $X$ is not bounded below, we write $\inf X = -\infty$. If $X$ is not bounded above, we write $\sup X = \infty$.

**Example 1.5.1** *Consider the three sets.* $A = (-3, 7]$, $B = \{1/n : n = 1, 2, 3, \cdots\}$, $C = \{x : x > 0$ *and* $x^2 > 3\}$.

**Example 1.5.2** *Show that* $\sup X = \infty$ *if and only if every* $b$ *in* $\mathbb{R}$ *there exists an* $x$ *in* $X$ *such that* $x > b$.

The following characterization of the supremum is easy to prove:

**Theorem 1.5.1** *Let* $X$ *be a set of real numbers and* $b^*$ *a real number. Then* $\sup X = b^*$ *if and only if (a)* $x \leq b^*$ *for all* $x$ *in* $X$ *and (b) for each* $\epsilon > 0$ *there exists an* $x$ *in* $X$ *such that* $x > b^* - \epsilon$

# Chapter 2

# Sequence of Real Numbers

A sequence is a function $k \mapsto x(k)$ whose domain is the set $\{1, 2, 3, \ldots\}$ of all positive numbers. The terms $x(1)$, $x(2), \cdots$ of the sequences are usually denoted by using subscripts: $x_1$, $x_2$, $\cdots$ . We shall us the notation $\{x_k\}_{k=1}^{\infty}$, or simply $\{x_k\}$ to indicate an arbitrary sequence of real numbers. A sequence $\{x_k\}$ of real numbers is said to be

1. increasing (or nondecreasing) if $x_k \leq x_{k+1}$ for $k = 1, 2, \cdots$

2. strictly increasing if $x_k > x_{k+1}$ for $k = 1, 2, \cdots$

3. decreasing (or non-increasing) if $x_k \geq x_{k+1}$ for $k = 1, 2, \cdots$

4. strictly decreasing if $x_k > x_{k+1}$ for $k = 1, 2, \cdots$

A sequence that is increasing or decreasing is called monotone.

**Example 2.0.1** *Decide whether or not the three sequences of real numbers whose general terms are given below are monotone*
*(a)* $x_k = 1 - 1/k$ *(b)* $y_k = (-1)^k$ *(c)* $z_k = \sqrt{k+1} - \sqrt{k}$

A sequence $\{x_k\}$ is said to converge to a number $x$ if $x_k$ becomes arbitrary close to $x$ for all sufficiently large $k$. We write $\lim_{k\to\infty} = x$ or $x_k \to x$ as $k \to \infty$. The precise definition of convergence is as follows:

**Definition 2.0.1** *The sequence* $\{x_k\}$ *converges to* $x$ *and we write*

$$\lim_{k\to\infty} x_k = x$$

*if for every* $\epsilon > 0$ *there exists a natural number* $N$ *such that* $|x_k - x| < \epsilon$ *for all* $k > N$. *The number* $x$ *is called the limit of the sequence* $\{x_k\}$. *A convergent sequence is one that converges to some number.*

Note that the limit of a convergent sequence is unique. A sequence that does not converge to any real number is said to diverge. In some cases we use the notation $\lim_{k\to\infty}$ even if the sequence $\{x_k\}$ is divergent: If for each number $M$ there exists a number $N$ such that $x_k \geq M$ for all natural number $k \geq N$, then we say that $x_k$ approaches $\infty$ and write $\lim_{k\to\infty} = \infty$. In the same way we write $\lim_{k\to\infty} = -\infty$ if for every number $M$ there exists a number $N$ such that $x_k \leq -M$ for all $k \geq N$.

A sequence $\{x_k\}$ is bounded if there exists a number $M$ such that $|x_k| \leq M$ for all $k = 1, 2, \cdots$. It is easy to see that every convergent sequence is bounded: If $x_k \to x$, then by the definition of convergence, only finitely many terms of the sequence can lie outside the interval $I = (x-1, x+1)$. The set $I$ is bounded and the finite set of points from the sequence that are not in $I$ is bounded, so $\{x_k\}$ must be bounded. On the other hand, is every bounded sequence convergent? No. For example, the sequence $\{y_k\} = \{(-1)^k\}$ in the Example above. Suppose, however, that the sequence is monotone as well as bounded. Then it is convergent.

**Theorem 2.0.1** *Every bounded monotone sequence is convergent.*

**Proof.** Suppose that $\{x_k\}$ is increasing and bounded. Let $b^*$ be the least upper bound of the set $X = \{x_k : k = 1, 2, \ldots\}$ and let $\epsilon$ be an arbitrary positive number. Then $b^* - \epsilon$ is not an upper bound of $X$, so there must be a term $x_N$ of the sequence for which $x_N > b^* - \epsilon$. Because the sequence is increasing, $b^* - \epsilon < x_N \leq x_k$ for all $k > N$. But the $x_k$ are all less than or equal to $b^*$, so $b^* - \epsilon < x_N \leq b^*$. Thus, for any $\epsilon > 0$, there exists a number $N$ such that $|x_k - b^*| < \epsilon$ for all $k > N$. Hence $\{x_k\}$ converges to $b^*$. If $\{x_k\}$ is decreasing and bounded, the argument is analogous. ■

**Example 2.0.2** *Consider the sequence $\{x_k\}$ defined by*

$$x_1 = \sqrt{2}, \qquad x_{k+1} = \sqrt{x_k + 2}, \qquad k = 1, 2, \ldots$$

*By using the Theorem above, prove that the sequence is convergent and find its limit. (Hint: Prove by induction that $x_k < 2$ for all $k$. Then, prove that the sequence is (strictly) increasing)*

The following rules are quite convenient in handling convergent sequences.

**Theorem 2.0.2** *Suppose that the sequences $\{x_k\}$ and $\{y_k\}$ converges to $x$ and $y$ respectively. Then:*

1. $\lim_{k\to\infty}(x_k + y_k) = x + y$

2. $\lim_{k\to\infty}(x_k - y_k) = x - y$

3. $\lim_{k\to\infty}(x_k \times y_k) = x \times y$

4. $\lim_{k\to\infty}(x_k / y_k) = x/y$, *assuming that* $y_k \neq 0$ *for all* $k$ *and* $y \neq 0$.

## 2.1 Subsequences

Let $\{x_k\}$ be a sequence. Consider a strictly increasing sequence of natural numbers

$$k_1 < k_2 < k_3 < \cdots$$

and form a new sequence $\{y_j\}_{j=1}^{\infty}$, where $y_j = x_{k_j}$ for $j = 1, 2, \ldots$. The sequence $\{y_j\}_j = \{x_{k_j}\}_j$ is called a subsequence of $\{x_k\}$. Because the sequence $\{k_j\}$ is strictly increasing, $k_j \geq j$ for all $j$. The terms of the subsequence are all present in the original one. In fact, a subsequence can be viewed as the result of removing some (possibly none) of the terms of the original sequence.

**Note 1** Some proofs involves *pairs* of sequences $\{x_k\}_{k=1}^{\infty}$ and $\{x_{k^j}\}_{j=1}^{\infty}$ where $k^j \geq j$ for all $j$ but where the sequence $k^1, k^2, \ldots$ is not necessarily strictly increasing. Thus $\{x_{k^j}\}$ is not quite a subsequence of $\{x_k\}$. However, it is always possible to select terms from $\{x_{k^j}\}$ in such a way that we get a subsequence $\{x_{k_i}\}_i$ of $\{x_k\}_k$: Let $k_1 = k^1$ and generally $k_{i+1} = k^{k_i+1}$. Then $k_{i+1} \geq k_i + 1 > k_i$.

**Theorem 2.1.1** *Every subsequence of a convergent sequence is itself convergent and has the same limit as the original sequence.*

**Theorem 2.1.2** *If the sequence* $\{x_k\}$ *is bounded, then it contains a convergent subsequence.*

**Proof.** Suppose that $|x_k| \leq M$ for all $k = 1, 2, \ldots$. Let $y_n = \sup\{x_k : k \geq n\}$ for $n = 1, 2, \ldots$. Then $\{y_n\}$ is a decreasing sequence because *the* set $\{x_k : k \geq n\}$ shrinks as $n$ increases. The sequence is also bounded because $|y_n| \leq M$. According to the Theorem above, the sequence $\{y_n\}$ has a limit $x = \lim_{n\to\infty} y_n \in [-M, M]$. By the definition of $y_n$, we can choose a term $x_{k^n}$ from the original sequence $\{x_k\}$ (with $k^n \geq n$) satisfying $|y_n - x_{k^n}| < 1/n$ (see Theorem 5.1). Then,

$$|x - x_{k^n}| = |x - y_n + y_n - x_{k^n}| \leq |x - y_n| + |y_n - x_{k^n}| \leq |x - y_n| + 1/n$$

This shows that $x_{k^n} \to x$ as $n \to \infty$. By using the construction in Note 1, we can extract, from $\{x_k\}$, a subsequence of $\{x_{k^n}\}$ that converges to $x$. ■

## 2.2 Cauchy Sequence

Definition 5.1 for a convergent sequence uses the value of the limit. If the limit is unknown or inconvenient to calculate, the definition is not very useful because one cannot test all numbers to see if they meet the criterion. An important alternative necessary and sufficient condition for convergence is based on the following concept

**Definition 2.2.1** *A sequence $\{x_k\}$ of real number is called a Cauchy sequence if for every $\epsilon > 0$, there exists a natural number such that*

$$|x_n - x_m| < \epsilon \text{ for all } n > N \text{ and all } m > N$$

All the terms of a convergent sequence eventually cluster around the limit, so the sequence is a Cauchy sequence. The converse is also true, that is, every Cauchy sequence is convergent. Therefore, when the limit of a sequence is not convenient to calculate, one can determine whether a sequence is convergent by checking if it is a Cauchy sequence

**Theorem 2.2.1** *A sequence is convergent if and only if it is a Cauchy sequence.*

**Proof.** To prove the "only if part", suppose that $\{x_k\}$ converges to $x$. Given $\epsilon > 0$, choose a natural number $N$ such that $|x_k - x| < \epsilon/2$ for all $k > N$. Then, for $k > N$ and $m > N$,

$$|x_k - x_m| = |x_k - x + x - x_m| \leq |x_k - x| + |x - x_m| < \epsilon/2 + \epsilon/2 = \epsilon$$

Therefore, $\{x_k\}$ is a Cauchy sequence.

To prove the "if part", suppose that $\{x_k\}$ is a Cauchy sequence. We first show that the sequence is bounded. By the Cauchy property, there is a number $M$ such that $|x_k - x_M| < 1$ for $k > M$. This means that all points $x_k$ with $k > M$ have a distance from $x_M$ that is less than 1. Moreover, the finite set $\{x_1, x_2, \ldots, x_{M-1}\}$ is surely bounded. Hence $\{x_k\}$ is bounded. By Theorem 6.2, it has a convergent subsequence $\{x_{k_j}\}$. Let $x = \lim_j x_{k_j}$. Because $\{x_k\}$ is a Cauchy sequence, for every $\epsilon > 0$ there is a number $N$ such that $|x_n - x_m| < \epsilon/2$ for $n > N$ and $m > N$. Moreover, if $J$ is sufficiently large, $|x_{k_j} - x| < \epsilon/2$ for all $j > J$. Then for $k > N$ and $j > \max\{N, J\}$,

$$|x_k - x| \leq |x_k - x_{k_j}| + |x_{k_j} - x| < \epsilon/2 + \epsilon/2 = \epsilon$$

Hence $x_k \to x$ as $k \to \infty$. ■

**Example 2.2.1** *Prove that the sequence $\{x_k\}$ with the general term $x_k = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2}$ is a Cauchy sequence.*

### 2.2.1 Upper and Lower Limits.

Let $\{x_k\}$ be a sequence that is bounded above, and define $y_n = \sup\{x_k : k \geq n\}$ for $n = 1, 2, \ldots$. Each $y_n$ is a finite number and $\{y_n\}_n$ is a decreasing seq*uence*. Then, either $\lim_{n\to\infty} y_n$ exists or is $-\infty$. We call this limit the upper limit (or lim sup) of the sequence $\{x_k\}$ and we introduce the following notation:

$$\lim_{k\to\infty} \sup x_k = \lim_{n\to\infty} (\sup\{x_k : k \geq n\})$$

If $\{x_k\}$ is not bounded above, we write $\limsup_{k\to\infty} x_k = \infty$. Similarly, if $\{x_k\}$ is bounded below, its lower limit (or lim inf), is defined as

$$\lim_{k\to\infty} \inf x_k = \lim_{n\to\infty} (\inf\{x_k : k \geq n\})$$

If $\{x_k\}$ is not bounded below, we write $\liminf x_k = -\infty$. The symbols of lim sup and lim inf are often written as $\overline{\lim}$ and $\underline{\lim}$.

**Example 2.2.2** *Determine the $\overline{\lim}$ and $\underline{\lim}$ of the following sequences*

$$\text{(a) } \{x_k\} = \{(-1)^k\} \qquad \text{(b) } \{x_k\} = \left\{(-1)^k \left(2 + \frac{1}{k}\right) + 1\right\}$$

It is not difficult to see that $\underline{\lim}_{k\to\infty} x_k \leq \overline{\lim}_{k\to\infty} x_k$ for every sequence $\{x_k\}$. The following result is also rather easy.

**Theorem 2.2.2** *If the sequence $\{x_k\}$ is convergent, then*

$$\underline{\lim}_{k\to\infty} x_k = \overline{\lim}_{k\to\infty} x_k = \lim_{k\to\infty} x_k$$

*On the other hand, if $\underline{\lim}_{k\to\infty} x_k = \overline{\lim}_{k\to\infty} x_k$, then $\{x_k\}$ is convergent.*

## 2.3 Infimum and Supremum of Functions

Suppose that $f(x)$ is defined for all $x$ in $B$, where $B \subseteq \mathbb{R}^n$. We define the infimum and supremum of the function $f$ over $B$ by

$$\inf_{x \in B} f(x) = \inf\{f(x) : x \in B\} \text{ and } \sup_{x \in B} f(x) = \sup\{f(x) : x \in B\}$$

If a function defined over a set $B$, if $\inf_{x \in B} f(x) = y$ and if there exists a $c$ in $B$ such that $f(c) = y$, then we say that the infimum is attained at the point $c$ in $B$. In this case, the infimum $y$ is called the minimum of $f$ over $B$, and we often write min instead of inf. In the same way, we write max instead of sup when the supremum of $f$ over $B$ is attained in $B$, so becomes the maximum.

# Chapter 3

# Euclidean Space

Topology is the study of fundamental properties of sets and mappings. In this chapter, we introduce a few basic topological ideas and use them to establish some important results about sets, and about continuous functions from one set to another. We begin by describing the notion of a metric and a metric space in general. A metric on a set $X$ is simply a measure of distance and it is a function $d : X \times X \to \mathbb{R}$ satisfying the following four properties:

1. Positivity: $d(x, y) \geq 0$ and $d(x, x) = 0$ for all $x, y \in X$
2. Discrimination: $d(x, y) = 0$ implies that $x = y$.
3. Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in X$
4. The triangle inequality: $d(x, y) \leq d(x, z) + d(z, y)\ for$ all $x, y, z \in X$

If $d$ is a metric on a set $X$, then $(X, d)$ is called a metric space. Although many of the ideas discussed here may be generalized to arbitrary types of sets, we confine ourselves to considering sets in $\mathbb{R}^n$, that is, $X = \mathbb{R}^n$. The Euclidean metric $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined as, for any two vectors $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ in $\mathbb{R}^n$

$$d(x, y) \equiv \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

$(\mathbb{R}^n, d)$ is called the Euclidean space, where $d$ is the Euclidean metric. For any two points $x$ and $y$ in $\mathbb{R}^n$, $d(x, y)$ measures the Euclidean distance between $x$ and $y$ and it is called the norm of the vector difference between $x$ and $y$ ($d(x, y) = ||x - y||$)

**Example 3.0.1** *Show that if the points $x$ and $y$ in $\mathbb{R}^n$ satisfy $d(x,y) < r$, then $-r < x_j - y_j < r$ for all $j = 1, 2, \dots, n$.*

## 3.1 Convex Sets

Convexity plays an important role in theoretical economics. Let $x$ and $y$ be any two points in $\mathbb{R}^n$. The closed line segment between $x$ and $y$ is the set

$$[x, y] = \{z : \exists \lambda \in [0,1] \text{ s.t. } z = \lambda x + (1-\lambda)y\}$$

whose members are the convex combinations $z = \lambda x + (1-\lambda)y$, with $0 \leq \lambda \leq 1$, of the two points $x$ and $y$. The definition of a convex set in $\mathbb{R}^n$ is now easy to formulate.

**Definition 3.1.1** *A set $S$ in $\mathbb{R}^n$ is called convex if $[x, y] \subseteq S$ for all $x, y$ in $S$ or equivalently, if*

$$\lambda x + (1-\lambda)y \in S \textit{ for all } x, y \textit{ in } S \textit{ and all } \lambda \textit{ in } [0,1]$$

Note in particular that the empty set and also any set consisting of one single point are convex. Intuitively speaking, a convex set must be connected without any holes and its boundary must not be bent inwards at any point.

If $S$ and $T$ are two convex sets, then their intersection $S \cap T$ is also convex. More generally, if $S_1, \dots, S_m$ are convex sets in $\mathbb{R}^n$, then $S_1 \cap \dots \cap S_m$ is convex.

## 3.2 Open Sets and Closed Sets

In the Euclidean space, we can define the open ball around $a$ with radius $r$ for any $a$ in $\mathbb{R}^n$ and any $r > 0$. The open ball is denoted by

$$B_r(a) = \{x \in \mathbb{R}^n : d(a, x) < r\}$$

Let $S$ be an arbitrary subset of $\mathbb{R}^n$. A point $a$ in $S$ is called an interior point of $S$ if there is an open ball $B_r(a)$ centered at $a$ which lies entirely within $S$. Thus, an interior point of $S$ is completely surrounded by other points of $S$. The set of all interior points of $S$ is called the interior of $S$, and is denoted by $int(S)$. A set $S$ is called a neighborhood of $a$ if $a$ is an interior point of $S$, that is, if $S$ contains some open ball $B_r(a)$ around $a$.

**Definition 3.2.1** *A set $S$ in $\mathbb{R}^n$ is called open if all its members are interior points.*

Some important properties of open sets are summarized in the following theorem.

**Theorem 3.2.1** *1. The whole space $\mathbb{R}^n$ and the empty set $\emptyset$ are both open.*
*2. Arbitrary unions of open sets are open.*
*3. The intersection of finitely many open sets is open.*

**Proof.** 1. It is clear that $B_1(a) \subseteq \mathbb{R}^n$ for all $a$ in $\mathbb{R}^n$, so $\mathbb{R}^n$ is open. The empty set $\emptyset$ is open because the set has no element, so every member is an interior point.

2. Let $\{U_i\}_{i \in I}$ be an arbitrary family of open sets in $\mathbb{R}^n$ and $U^* = \cup_{i \in I} U_i$ be the union of the whole family. For each $x$ in $U^*$, there is at least one $i$ in $I$ such that $x \in U_i$. Since $U_i$ is open, there exists an open ball $B_r(x)$ with center $x$ such that $B_r(x) \subseteq U_i \subseteq U^*$. Hence, $x$ is an interior point of $U^*$. This shows that $U^*$ is open.

3. Let $\{U_i\}_{i=1}^m$ be a finite collection of open sets in $\mathbb{R}^n$ and $U_* = \cap_{i=1}^m U_i$ be the intersection of all these sets. Let $x$ be any point in $U_*$. Then for each $i = 1, \cdots, m$, the point $x$ belongs to $U_i$, and because $U_i$ is open, there exists an open ball $B_i = B_{r_i}(x)$ with center $x$ and radius $r_i > 0$ such that $B_i \subseteq U_i$. Let $B_* = B_r(x)$ where $r$ is the smallest of the numbers $r_1, \ldots, r_m$. Then $x \in B_* = \cap_{i=1}^m B_i \subseteq \cap_{i=1}^m U_i = U_*$ and it follows that $U_*$ is open. ■

Note that the intersection of an infinite number of open sets need not be open. For example, the intersection of the infinite family $B_{1/k}(0)$, $k = 1, 2, \cdots$, of open balls center at the zero vector 0 is the one-element set $\{0\}$. The set $\{0\}$ is not open because $B_r(0)$ is not a subset of $\{0\}$ for any positive $r$.

**Example 3.2.1** *Show that $A = \{(x, y) : x > y\}$ is an open set in $\mathbb{R}^2$.*

The complement of a set $S \subseteq \mathbb{R}^n$ is the set $S^c$ of all points in $\mathbb{R}^n$ that do not belong to $S$. A point $x$ in $\mathbb{R}^n$ is called a boundary point of the set $S$ if every ball centered at $x$ contains at least one point in $S$ and at least one point in $S^c$. Note that a boundary point of $S$ is also a boundary point of $S^c$, and vice versa. Each point in a set is either an interior point or a boundary point of the set. Note that given any set $S \subseteq \mathbb{R}^n$, there is a corresponding partition of $\mathbb{R}^n$ into three mutually disjoint sets (some of which may be empty), namely:

1. the interior of $S$, which consists of all point $x$ in $\mathbb{R}^n$ such that $N \subseteq S$ for some neighborhood $N$ of $x$.

2. the exterior of $S$, which consists of all points $x$ in $\mathbb{R}^n$ for which there exists some neighborhood $N$ of $x$ such that $N \subseteq S^c$.

3. the boundary of $S$, which consists of all points $x$ in $\mathbb{R}^n$ with the property that every neighborhood $N$ of $x$ intersects both $S$ and its complement $S^c$

The set of all boundary points of a set $S$ is called the boundary of $S$ and is denoted by $\partial S$ or $bd(S)$. A set $S$ in $\mathbb{R}^n$ is said to be closed if it contains all its boundary points. The union $S \cup \partial S$ is called the closure of $S$, denoted by $\overline{S}$ or $c\ell(S)$. A point $a$ belongs to $\overline{S}$ if and only if every open ball $B_r(a)$ around $a$ intersects $S$. The closure $\overline{S}$ of any set $S$ is indeed closed. In fact, $\overline{S}$ is the smallest closed set containing $S$.

We noted above that $S$ and $S^c$ have the same boundary points. Furthermore, a set is open if and only if every point in the set is an interior point, that is, if and only if it contains none of its boundary points. On the other hand, a set is closed if and only if it contains all its boundary points. It is easily follows the following statement is true.

A set in $\mathbb{R}^n$ is closed if and only if its complement is open.

Here are most important properties of closed sets.

**Theorem 3.2.2** *1. The whole space $\mathbb{R}^n$ and the empty set are both closed.*
*2. Arbitrary intersections of closed sets are closed.*
*3. The union of finitely many closed sets is closed.*

**Example 3.2.2** *Sketch the set $S = \{(x, y) \in \mathbb{R}^2 : x > 0, y \geq 1/x\}$ in the plane. Is $S$ closed?*

Note that infinite unions of closed sets need not be closed. Also, one should be careful to note that the technical meaning of the words open and closed. In topology, any set containing some of its boundary points but not all of them is neither open nor closed. For example, half-open intervals $[a, b)$ and $(a, b]$ are neither open nor closed in $\mathbb{R}$. By contrast, the empty set,$\emptyset$, and the whole space $\mathbb{R}^n$ are both open and closed. These are the only two sets in $\mathbb{R}^n$ that are both open and closed.

## 3.3 Topology and Convergence

We generalize the argument in Chapter 2 into $\mathbb{R}^n$. A sequence $\{x_k\}$ (alternatively, denoted by $\{x_k\}_{k=1}^{\infty}$ or $\{x_k\}_k$) in $\mathbb{R}^n$ is a function that for each natural number $k$ yields a corresponding point $x_k$ in $\mathbb{R}^n$. The point $x_k$ os called the $k$th term or $k$th element of the sequence.

**Definition 3.3.1** *A sequence $\{x_k\}$ in $\mathbb{R}^n$ converges to a point $x$ if for each $\epsilon > 0$, there exists a natural number $N$ such that $x_k \in B_\epsilon(x)$ for all $k > N$, or equivalently, if $d(x_k, x) \to 0$ as $k \to \infty$.*

In other words, each open ball around $x$, however small its radius $\epsilon$, must contain $x_k$ for all sufficiently large $k$. Geometrically speaking, as $k$ increases, the points $x_k$ must eventually all become concentrated around $x$. Note that $x_k$ need not approach $x$ from any fixed direction and the distance $d(x_k, x)$ need not decrease monotonically as $k$ increases. If $\{x_k\}$ converges to $x$, we write

$$x_k \to x \text{ as } k \to \infty, \qquad \text{or} \qquad \lim_{k\to\infty} x_k = x$$

and call $x$ the limit of the sequence.

**Theorem 3.3.1** *Let $\{x_k\}$ be a sequence in $\mathbb{R}^n$. Then $\{x_k\}$ converges to the vector $x$ in $\mathbb{R}^n$ if and only if for each $j = 1, \ldots, n$, the real number sequence $\{x_k^{(j)}\}_{k=1}^{\infty}$ consisting of the $j$th component of each vector $x_k$ converges to $x^{(j)}$, the $j$th component of $x$.*

**Proof.** For every $k$ and every $j$ one has $d(x_k, x) = ||x_k - x|| \geq |x_k^{(j)} - x^{(j)}|$. It follows that if $x_k \to x$, then $x_k^{(j)} \to x^{(j)}$.

Suppose on the other hand that $x_k^{(j)} \to x^{(j)}$ for $j = 1, 2, \ldots, n$. Then, given any $\epsilon > 0$, for each $j = 1, 2, \ldots, n$ there exists a number $N_j$ such that $|x_k^{(j)} - x^{(j)}| < \epsilon/\sqrt{n}$ for all $k > N_j$. It follows that

$$d(x_k, x) = \sqrt{|x_k^{(1)} - x^{(1)}|^2 + \cdots + |x_k^{(n)} - x^{(n)}|^2}$$
$$< \sqrt{\epsilon^2/n + \cdots + \epsilon^2/n} = \sqrt{\epsilon^2} = \epsilon$$

for all $k > \max\{N_1, \ldots, N_n\}$. Therefore, $x_k \to x$ as $k \to \infty$. ■

This characterization makes it easy to translate theorems about sequences of real numbers into theorems about sequences in $\mathbb{R}^n$.

Let $\{x_k\}$ be a sequence in $\mathbb{R}^n$. Consider a strictly increasing sequence $k_1 < k_2 < k_3 < \cdots$ of natural numbers and let $y_j = x_{k_j}$ for $j = 1, 2, \ldots$. The sequence $\{y_j\}_{j=1}^{\infty}$ is called a subsequence of $\{x_k\}$ and is often denoted by $\{x_{k_j}\}_{j=1}^{\infty}$. All terms of the subsequence $\{x_{k_j}\}_j$ are present in the original sequence $\{x_k\}_k$ but some or even most terms of the original sequence may be omitted as long as infinitely many remain.

**Example 3.3.1** *Examine the convergence of the following sequences in* $\mathbb{R}^2$

1. $x_k = (1/k, 1 + 1/k)$
2. $x_k = (1 + 1/k, (1 + 1/k)^k)$
3. $x_k = (k, 1 + 3/k)$
4. $x_k = ((k + 2)/3k, (-1)^k/2k)$

### 3.3.1 Cauchy Sequences

Cauchy sequences of real numbers are studied in Chapter 2. There is a natural generalization to $\mathbb{R}^n$.

**Definition 3.3.2** *A sequence* $\{x_k\}$ *in* $\mathbb{R}^n$ *is called a Cauchy sequence if for every* $\epsilon > 0$ *there exists a number* $N$ *such that* $d(x_k, x_m) < \epsilon$ *for all* $k > N$ *and all* $m > N$.

The main results in Chapter 2 on Cauchy sequences in $\mathbb{R}$ carry over without difficulty to sequences in $\mathbb{R}^n$. In particular,

**Theorem 3.3.2** *A sequence* $\{x_k\}$ *in* $\mathbb{R}^n$ *is convergent if and only if it is a Cauchy sequence.*

**Example 3.3.2** *Prove Theorem 11.2.*

Convergent sequences can be used to characterize very simply the closure of any set in $\mathbb{R}^n$.

**Theorem 3.3.3** *1. For any set* $S \subseteq \mathbb{R}^n$, *a point* $a$ *in* $\mathbb{R}^n$ *belongs to* $\overline{S}$ *if and only if* $a$ *is the limit of a sequence* $\{x_k\}$ *in* $S$.

*2. A set* $S \subseteq \mathbb{R}^n$ *is closed if and only if every convergent sequence of points in* $S$ *has its limit in* $S$.

**Proof.** 1. Let $a \in \overline{S}$. For each natural number $k$, the open ball $B_{1/k}(a)$ must intersect $S$, we can choose an $x_k$ in $B_{1/k}(a) \cap S$. Then $x_k \to a$ as $k \to \infty$. On the other hand, assume that $a = \lim_{k\to\infty} x_k$ for some sequence $\{x_k\}$ in $S$. We claim that $a \in \overline{S}$. For any $r > 0$, we know that $x_k \in B_r(a)$ for all large enough $k$. Since $x_k$ also belongs to $S$, it follows that $B_r(a) \cap S \neq \emptyset$. Hence $a \in \overline{S}$.

2. Assume that $S$ is closed and *let* $\{x_k\}$ be a convergent sequence such that $x_k \in S$ for all $k$. By part 1, $x = \lim_{k\to\infty} x_k$ belongs to $\overline{S} = S$. Conversely, suppose that every sequence of points from $S$ has its limit in $S$. Let $a$ be a point in $\overline{S}$. By part 1, $a = \lim_{k\to\infty} x_k$ for some sequence $x_k$ in $S$ and there fore $a \in S$, by hypothesis. This shows that $\overline{S} \subseteq S$, hence $S$ is closed. ■

### 3.3.2 Boundedness in $\mathbb{R}^n$

**Definition 3.3.3** *A set $S$ in $\mathbb{R}^n$ is bounded if there exists a number $M$ such that $||x|| \leq M$ for all $x$ in $S$. In other words, no point of $S$ is at a distance greater than $M$ from the origin. A set that is not bounded is called unbounded.*

Similarly, a sequence $\{x_k\}$ in $\mathbb{R}^n$ is bounded if the set $\{x_k : k = 1, 2, \ldots\}$ is bounded.

**Theorem 3.3.4** *Every convergent sequence in $\mathbb{R}^n$ is bounded.*

**Proof.** If $x_k \to x$, then only finitely many terms of the sequence can lie outside the ball $B_1(x)$. The ball $B_1(x)$ is bounded and any finite set of points is bounded, so $\{x_k\}$ must be bounded. ■

Even though a bounded sequence in $\mathbb{R}^n$ is not necessarily convergent, any bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

**Theorem 3.3.5** *Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

Finally, we can fully characterize a bounded subset $S$ of $\mathbb{R}^n$ as follows.

**Theorem 3.3.6** *A subset $S$ of $\mathbb{R}^n$ is bounded if and only if every sequence of points in $S$ has a convergent subsequence.*

### 3.3.3 Compactness

**Definition 3.3.4** *A subset $S$ of $\mathbb{R}^n$ is compact if it is closed and bounded.*

Compactness is a central concept in mathematical analysis. It also plays an important role in mathematical economics, for example, in proving existence of solutions to maximization problems. Compact sets in $\mathbb{R}^n$ can be given the following very useful characterization.

**Theorem 3.3.7** *A subset $S$ of $\mathbb{R}^n$ is compact if and only if every sequence of points in $S$ has a subsequence that converges to a point in $S$.*

**Proof.** Suppose that $S$ is compact and let $\{x_k\}$ be a sequence in $S$. By Theorem 11.6, $\{x_k\}$ contains a convergent subsequence. Since $S$ is closed, it follows from Theorem 11.3 that the limit of the subsequence must be in $S$.

On the other hand, suppose that every sequence of points in $S$ has a subsequence converging to a point of $S$. We must prove that $S$ is closed and bounded. Boundedness follows from Theorem 11.6. To prove that $S$ is closed, let $x$ be any point in its closure $\overline{S}$. By Theorem 11.3, there is a subsequence $\{x_{k_j}\}$ that converges to a limit $x'$ in $S$. But $\{x_{k_j}\}$ also converges to $x$. Hence $x = x' \in S$. ∎

## 3.4 Continuous Functions

Consider first a real-valued function $z = f(x) = f(x_1, \ldots, x_n)$of $n$ variables. Roughly speaking, $f$ is continuous if small changes in the independent variables cause only small changes in the function value. The precise "$\epsilon - \delta$" definition is as follows.

**Definition 3.4.1** *A function $f$ with domain $S \subseteq \mathbb{R}^n$is continuous at a point $a$ in $S$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$|f(x) - f(a)| < \epsilon \text{ for all } x \text{ in } S \text{ with } ||x - a|| < \delta$$

*If $f$ is continuous at every point $a$ in a set $S$, we say that $f$ is continuous on $S$.*

**Example 3.4.1** *Let $f(x) = \sqrt{x}$ be a function from $\mathbb{R}_+$ to $\mathbb{R}$. Prove that $f$ is continuous.*

A function of $n$ variables that can be constructed from continuous functions by combining the operations of addition, subtraction, multiplication, division, and composition of functions, is continuous wherever it is defined. Now consider the general case of vector-valued functions.

**Definition 3.4.2** *A function $f = (f_1, \dots, f_m)$ from a subset $S$ of $\mathbb{R}^n$ to $\mathbb{R}^m$ is said to be continuous at $x^\circ$ in $S$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $d(f(x), f(x^\circ)) < \epsilon$ for all $x$ in $S$ with $d(x, x^\circ) < \delta$, or equivalently, such that $f(B_\delta(x^\circ) \cap S) \subseteq B_\epsilon(f(x^\circ))$.*

Intuitively, continuity of $f$ at $x^\circ$ means that $f(x)$ is close to $f(x^\circ)$ when $x$ is close to $x^\circ$. Frequently, the easiest way to show that a vector function is continuous, is to show that each component is continuous.

**Theorem 3.4.1** *A function $f = (f_1, \dots, f_m)$ from $S \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$ is continuous at a point $x^\circ$ in $S$ if and only if each component $f_j : S \to \mathbb{R}$, $j = 1, \dots, m$, is continuous at $x^\circ$.*

**Proof.** Suppose that $f$ is continuous at $x^\circ$. Then, for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f_j(x) - f_j(x^\circ)| \leq d(f(x), f(x^\circ)) < \epsilon$$

for every $x$ in $S$ with $d(x, x^\circ) < \delta$. Hence $f_j$ is continuous at $x^\circ$ for $j = 1, \dots, m$.

Suppose on the other hand that each component $f_j$ is continuous at $x^\circ$. Then, for every $\epsilon > 0$ and every $j = 1, \dots, m$, there exists a $\delta_j > 0$ such that $|f_j(x) - f(x^\circ)| < \epsilon/\sqrt{m}$ for every point $x$ in $S$ with $d(x, x^\circ) < \delta_j$. Let $\delta = \min\{\delta_1, \cdots \delta_m\}$. Then, $x \in B_\delta(x^\circ) \cap S$ implies that

$$\begin{aligned} d(f(x), f(x^\circ)) &= \sqrt{|f_1(x) - f_1(x^\circ)|^2 + \cdots + |f_m(x) - f_m(x^\circ)|^2} \\ &< \sqrt{\epsilon^2/m + \cdots \epsilon^2/m} = \epsilon \end{aligned}$$

This proves that $f$ is continuous at $x^\circ$. ■

Continuity of a function can be characterized by means of convergent sequences. In theoretical arguments, this is often the easiest way to check if a function is continuous.

**Theorem 3.4.2** *A function $f$ from $S \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$ is continuous at a point $x^\circ$ in $S$ if and only if $f(x_k) \to f(x^\circ)$ for every sequence $\{x_k\}$ of points in $S$ that converges to $x^\circ$.*

**Proof.** Suppose that $f$ is continuous at $x^\circ$, and let $\{x_k\}$ be a sequence in $S$ that converges to $x^\circ$. Let $\epsilon > 0$ be given. Then there exists a $\delta > 0$ such that $d(f(x), f(x^\circ)) < \epsilon$ whenever $x \in B_\delta(x^\circ) \cap S$. Because $x_k \to x^\circ$, there exists a number $N$ such that $d(x_k, x^\circ) < \delta$ for all $k > N$. But then $x_k \in B_\delta(x^\circ) \cap S$ and so $d(f(x_k), f(x^\circ)) < \epsilon$ for all $k > N$, which implies that $\{f(x_k)\}$ converges to $f(x^\circ)$.

On the other hand, let $\{x_k\}$ be a sequence such that $x_k \in S$ for each $k$ and it converges to $x^\circ$. Since the sequence converges to $x^\circ$, for any $\delta > 0$ there exists a number $N_\delta$ such that $d(x_k, x^\circ) < \delta$ for all $k > N_\delta$. Similarly, since $f(x_k) \to f(x^\circ)$, for any $\epsilon > 0$ there exists a number $N_\epsilon$ such that $d(f(x_k), f(x^\circ)) < \epsilon$ for all $k > N_\epsilon$. Define $N^* = \max\{N_\delta, N_\epsilon\}$. Then, by choosing $k > N^*$ for any $\epsilon > 0$ there exists a $\delta > 0$ with $d(x_k, x^\circ) < \delta$ such that $d(f(x_k), f(x^\circ)) < \epsilon$. This proves that $f$ is continuous at $x^\circ$. ■

The following property of continuous functions is often useful.

**Theorem 3.4.3** *Let $S \subseteq \mathbb{R}^n$ and let $f : S \to \mathbb{R}^m$ be continuous. Then, $f(K) = \{f(x) : x \in K\}$ is compact for every compact subset $K$ of $S$.*

**Proof.** Let $\{y_k\}$ be any sequence in $f(K)$. By definition, for each $k$ there is a point $x_k$ in $K$ such that $y_k = f(x_k)$. Because $K$ is compact, the sequence $\{x_k\}$ has a subsequence $\{x_{k_j}\}$ converging to a point $x_0$ in $K$ by Theorem 11.7. Because $f$ is continuous, $f(x_{k_j}) \to f(x_0)$ as $j \to \infty$, where $f(x_0) \in f(K)$ because $x_0 \in K$. But then $\{y_{k_j}\}$ is a subsequence of $\{y_k\}$ that converges to a point $f(x_0)$ in $f(K)$. So, we have proved that any sequence in $f(K)$ has a subsequence converging to a point of $f(K)$. By Theorem 11.7, it follows that $f(K)$ is compact. ■

Suppose that $f$ is a continuous function from $\mathbb{R}^n$ to $\mathbb{R}^m$. If $V$ is an open set in $\mathbb{R}^n$, the image $f(V) = \{f(x) : x \in V\}$ of $V$ need not be open in $\mathbb{R}^m$. Nor need $f(C)$ be closed if $C$ is closed. Nevertheless, the inverse image (or preimage) $f^{-1}(U) = \{x : f(x) \in U\}$ of an open set $U$ under a continuous function $f$ is always open. Similarly, the inverse image of any closed set must be closed.

**Theorem 3.4.4** *Let $f$ be any function from $\mathbb{R}^n$ to $\mathbb{R}^m$. Then, $f$ is continuous if and only if either of the following equivalent conditions is satisfied:*

1. *$f^{-1}(U)$ is open for each open set $U$ in $\mathbb{R}^m$*
2. *$f^{-1}(F)$ is closed for each closed set $F$ in $\mathbb{R}^m$*

**Example 3.4.2** *Give examples of subsets $S$ of $\mathbb{R}$ and continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that*

1. *$S$ is closed but $f(S)$ is not closed*
2. *$S$ is open but $f(S)$ is not open*
3. *$S$ is bounded but $f(S)$ is not bounded*

## 3.5 Some Existence Theorems

We consider some very powerful topological results, each with important application in microeconomic theory. All are in a class of theorems known as "existence theorems." An existence theorem specifies conditions that, if met, guarantee that something exists.

Before presenting the first existence theorem known as Weierstrass Existence Theorem, we present the following theorem, which will be used in proving Weierstrass Existence Theorem.

**Theorem 3.5.1** *Let $S$ be a compact set in $\mathbb{R}$ and let $x_*$ be the greatest lower bound of $S$ and $x^*$ be the lowest upper bound of $S$. Then both $x_*$ and $x^*$ are in $S$.*

**Proof.** Let $S \subset \mathbb{R}$ be closed and bounded and let $x^*$ be the lowest upper bound of $S$. Then, by definition of any upper bound, we have $x^* \geq x$ for all $x \in S$. If $x^* = x$ for some $x \in S$, we are done. Suppose that $x^*$ is strictly greater than every point in $S$. If $x^* > x$ for all $x \in S$, then $x^* \notin S$, so $x^* \in \mathbb{R}\backslash S$. Since $S$ is closed, $\mathbb{R}\backslash S$ is open. Then, by the definition of open sets, there exists some $\epsilon > 0$ such that $B_\epsilon(x^*) = (x^* - \epsilon, x^* + \epsilon) \subset \mathbb{R}\backslash S$. Since $x^* > x$ for all $x \in S$ and $B_\epsilon(x^*) \subset \mathbb{R}\backslash S$, there exists $\widetilde{x} \in B_\epsilon(x^*)$ such that $\widetilde{x} > x$ for all $x \in S$. In particular, we have $x^* - \epsilon/2 \in B_\epsilon(x^*)$ and $x^* - \epsilon/2 > x$ for all $x \in S$. But, this contradicts that $x^*$ is the lowest upper bound of $S$. Therefore, we must conclude that $x^* \in S$. The same argument can be constructed for the greatest lower bound of $S$. ■

Weierstrass Existence Theorem that we consider is a fundamental result in optimization theory. Many problems in economics involves maximizing or minimizing a function defined over some subset of $\mathbb{R}^n$. We will pay particular attention to problems of maximizing or minimizing functions that map vectors in $\mathbb{R}^n$ to numbers of $\mathbb{R}$. The theorem specifies sufficient conditions under which the existence of a maximum and minimum of a function is assured.

**Theorem 3.5.2** *(Weierstrass Existence of Extreme Values) Let $f : S \to \mathbb{R}$ be a continuous function where $S$ is a non-empty compact subset of $\mathbb{R}^n$. Then, there exists a vector $\overline{x}$ and a vector $\underline{x}$ in $S$ such that*

$$f(\underline{x}) \leq f(x) \leq f(\overline{x}) \text{ for all } x \in S$$

**Proof.** Since $f$ is continuous and $S$ is compact we know by Theorem 12.3 that $f(S)$ is a compact set. Because $f$ is real-valued, $f(S) \subset \mathbb{R}$. Since $f(S)$ is compact, it is closed and bounded. By Theorem 13.1, any closed and bounded subset of real numbers contains its greatest lower bound, call it $a$, and its lowest upper bound, call it $b$. By the definition of the image set, there exist some $\overline{x} \in S$ such that $f(\overline{x}) = b \in f(S)$ and some $\underline{x} \in S$ such that $f(\underline{x}) = a \in f(S)$. Together with the definitions of the greatest lower bound and the lowest upper bound, we have $f(\underline{x}) \leq f(x)$ and $f(x) \leq f(\overline{x})$ for all $x \in S$. ■

Next, let us turn our attention to one more special class of functions. We restrict our attention to functions that map from one subset of $\mathbb{R}^n$ back to the same subset of $\mathbb{R}^n$. If $S \subseteq \mathbb{R}^n$ and $f : S \to S$, then $f$ maps vectors in $S$ back to other vectors in the same set $S$. For example, a system of simultaneous equations given by

$$\begin{aligned} y_1 &= f_1(x_1, \ldots, x_n) \\ &\vdots \\ y_n &= f_n(x_1, \ldots, x_n) \end{aligned}$$

maps a vector $(x_1, \ldots, x_n) \in \mathbb{R}^n$ into a vector $(y_1, \ldots, y_n) \in \mathbb{R}^n$. Many times we are interested in the solution to such systems. In some special cases, the solution will be some $(x_1^*, \ldots, x_n^*) \in \mathbb{R}^n$, where

$$\begin{aligned} x_1^* &= f_1(x_1^*, \ldots, x_n^*) \\ &\vdots \\ x_n^* &= f_n(x_1^*, \ldots, x_n^*) \end{aligned}$$

A vector $x^* = (x_1^*, \ldots, x_n^*)$ is called a fixed point of the function $f : \mathbb{R}^n \to \mathbb{R}^n$. The phrase "fixed point" is used here because, should such a point exist, it will be one that is left undisturbed, or unmoved by the mapping in going from the domain to the range.

Many profound questions about the fundamental consistency of microeconomic systems have been answered by reformulating the question as one of the existence of a fixed point. One very powerful theorem provides us with sufficient conditions under which a fixed point is guaranteed to exist.

**Theorem 3.5.3** *(Brouwer Fixed-Point Theorem) Let $S \subset \mathbb{R}^n$ be a non-empty compact and convex set. Let $f : S \to S$ be a continuous function. Then there exists at least one fixed point of $f$ in $S$. That is, there exists at least one $x^* \in S$ such that $x^* = f(x^*)$.*

Proof is quite technical and it can be found in Border (1985).

Now we consider some theorem of a geometric nature with many applications in economic theory. The main result states that two disjoint convex sets in $\mathbb{R}^n$ can be separated by a hyperplane. In two dimensions, hyperplanes are straight lines. With its simple geometrical interpretation, the separation theorem in $\mathbb{R}^n$ is one of the most fundamental tools in modern optimization theory and an early economic application of a separation theorem was to welfare economics.

If $a$ is a nonzero vector in $\mathbb{R}^n$ and $\alpha$ is a real number, then the set

$$H = \{x \in \mathbb{R}^n : a \cdot x = \alpha\}$$

is a hyperplane in $\mathbb{R}^n$ with $a$ as its normal. Moreover, the hyperplane $H$ separates $\mathbb{R}^n$ into two convex spaces:

$$\begin{aligned} H_+ &= \{x \in \mathbb{R}^n : a \cdot x \geq \alpha\} \\ H_- &= \{x \in \mathbb{R}^n : a \cdot x \leq \alpha\} \end{aligned}$$

If $S$ and $T$ are subsets of $\mathbb{R}^n$, then $H$ is said to separate $S$ and $T$ if $S$ is contained in one of $H_+$ and $H_-$ and $T$ is contained in the other. In other words, $S$ and $T$ can be separated by a hyperplane if there exist a nonzero vector $a$ and a scalar $\alpha$ such that

$$a \cdot x \leq \alpha \leq a \cdot y \text{ for all } x \text{ in } S \text{ and all } y \text{ in } T$$

**Theorem 3.5.4** *(Separating Hyperplane Theorem) Let $S$ and $T$ be two disjoint non-empty convex sets in $\mathbb{R}^n$. Then, there exists a non zero vector $a$ in $\mathbb{R}^n$ and a scalar $\alpha$ such that*

$$a \cdot x \leq \alpha \leq a \cdot y \textit{ for all } x \textit{ in } S \textit{ and all } y \textit{ in } T$$

*Thus, $S$ and $T$ are separated by the hyperplane $H = \{z \in \mathbb{R}^n : a \cdot z = \alpha\}$.*

# Bibliography

[1] "Fixed Point Theorems with Applications to Economic and Game Theory," by Kim C. Border, 1985, Cambridge University Press.

[2] "Further Mathematics for Economic Analysis," Prentice Hall, by Knut Sydsæter, Peter Hammond, Atle Seierstad, and Atle Strom, 2005, Prentice Hall

[3] "Mathematical Appendix," in Advanced Microeconomic Theory, Second Edition, by Geoffrey A. Jehle and Philip J. Reny, 2000, Addison Wesley